GeoGebra as facilitator of a hypotrochoid generalization in an initiation scientific research project at undergraduate level

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Abstract

An initiation to scientific research project was proposed for two undergraduate students in Environmental Engineering, with the porpuse of studing a generalization of the hypotrochoid and epitrochoid curves. The GeoGebra software is the facilitator of the team of two professors and the students. The difficulties of the mathematical proofs become, in the environment of the dynamic geometry, exercises of construction of curves that, arranged in sequence, determined the generalization of the well-known hypotrochoid curve, result that is the detailed in this work. From this first discovery, the transition to epitrochoid curves or more general curve,s becomes an iterative application and an applicattion of mathematical induction. This study has demonstrated the importance of the computational tool, specially for its contribution in the visualization of theoretical results as well as for its simplicity in coding.

Keywords: Hypotrochoid; undergraduate research in Geometry; GeoGebra.

1. Introduction

The research on a generalization of the hypotrochoid and epitrochoid curves [1] began to be carried out as part of a scientific initiation project, in which two undergraduate students in Environmental Engineering were encouraged to research the subject. It began with a bibliographic and intenet search, where the students found materials of a dynamic visualization that could be possible to reproduce using the software GeoGebra [2]. Motivated by these images, the first step in the research was to study the classical hypotrochoid curve and its implementation in GeoGebra. This process facilitated the discovery of a more general formula, starting the dynamic with three circles, from which the students began to create detailed routines that had not yet been presented in the literature.

The students began understanding the dynamics of the construction of the hypotrochoid in the most simple way, not appealing for rigoruous mathematical definitions. They understood that the hypotrochoid is a curve described by a point *P*, the pole, attached to a given circumference C_1, that rolls without slipping, along a second circumference C_0, that is fixed, as shown in Figure 1. The segment with extremes in the center of C_1 and the pole, we denote by d.

In history, according to Zbynek et al [3], the hipocycloide, which is a hipotrocoide where d is equal to the radius of the moving circumference, dates back to the Greeks, who used them to explain the retrograde motion of the planets. In the mid-1570s, the mathematician Girolamo Cardano (1501) described hypocycloid applications in high-speed printing press technology. In 1674, the Danish astronomer Ole Roemer began a systematic investigation of cycloid curves in connection with mechanical gears. Other mathematicians who studied these curves were, among others, Laurent La Hire (1606-1656), Gérard Desargues (1591-1661), Isaac Newton (1642-1726) and Leonhard Euler (1707-1783).

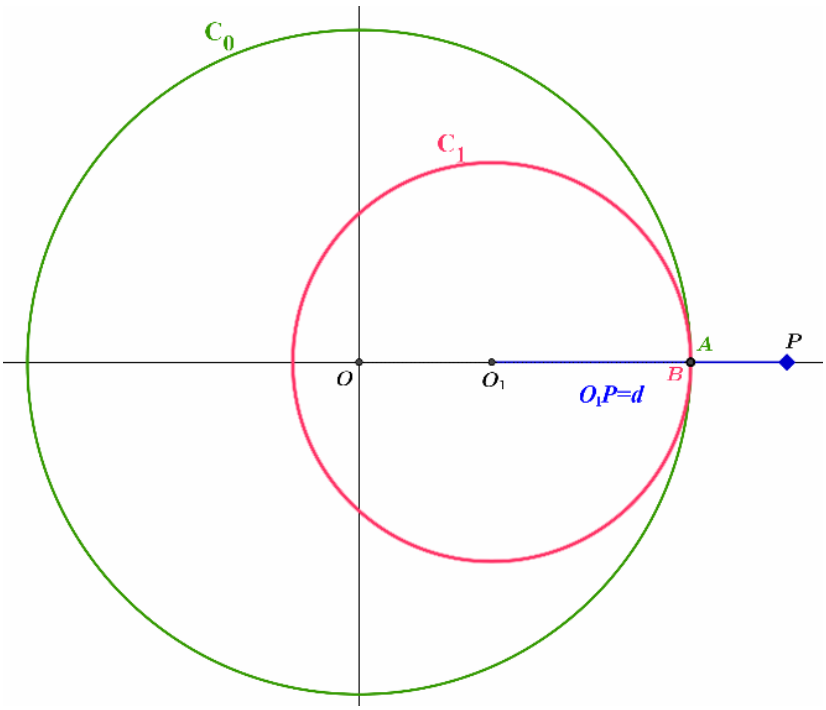


Figure 1. Hypotrochoid at time zero

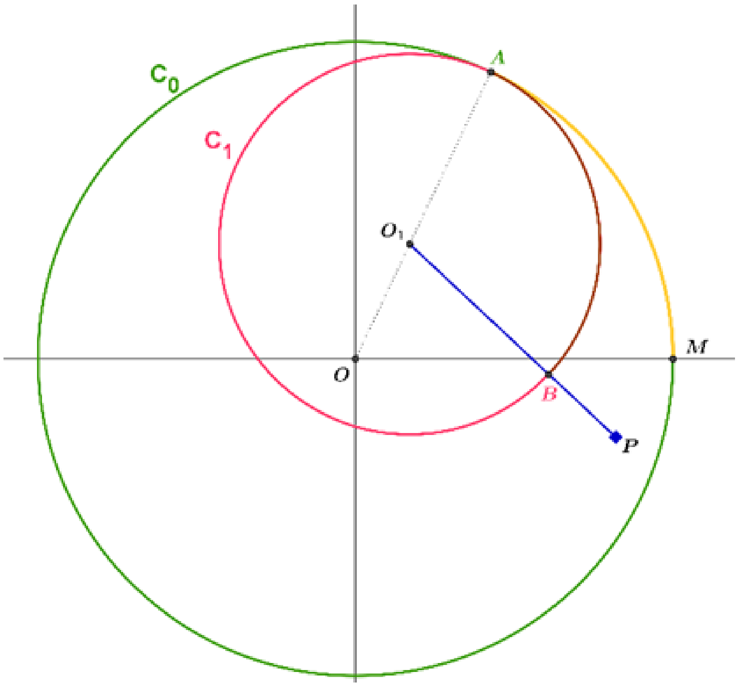


Figure 2. Hypotrochoid at time T≠0.

In 1800 the hipocicloide began to be studied for train gears in engineering. Many rotary machines traditionally use this type of curves; let us remember the hydraulic motors, clocks and timing devices, the Wankel motor, rotary pump piston, speed reducers, among others. Wankel's spinning engine was invented by the German engineer Felix Wankel, who received his first patent for the engine in 1929. Currently, the engine is used in automobiles, and the applications of these curves are still an active area of research.
This study is organize as follows: In section 1, the hypotrochoid general formula is obtained mathematically, discovery from the continuously feedback by the simulations in GeoGebra. In section 2, it is explained in detail the construction in GeoGebra of the first result of generalization of the hypotrochoid of three circles and a pole. Finalizing the study with simulations in GeoGebra and the final considerations.
We emphasize that all the figures of this work were obtained in GeoGebra by the authors.

2. Mathematical proof with GeoGebra's assistance: generalized hypotrochoid formula

This section starts with the first generalization of the hypotrochoid curve considering three circumferences: a fixed, C_0, of center O, a moving circumference, C_1, with center in O_1, tangent to C_0 that, at the first instant time, the angle MOA, denoted by $\angle$MOA, with vertex in O is spinning without slipping counterclockwise (see Figure 2); a third circumference, C_2, of center O_2, tangent to C_1 spinning clockwise $\angle AO_1B$. Finally, the pole P belonging to the ray starting in O_2 and determining the angle $\angle BO_2$P with the ray with origin in O_1, considered in clockwise sense. As hypothesis for this construction, it is considering the equality of the angles $\angle MOA$, $\angle AO_1$B and $\angle BO_2P$, with the objective of preserving the dynamic of the classical hypotrochoid, described in the introduction.
Starting with the proof , let us denote OA = R, O_1A = r_1, O_2B = r_2, and O_2P = d. Observing Figure 2,

being OE the x-coordinate of O_1, and OF the x-coordinate of O_2, we have that the x-coordinate of P, denoted by $X(P)$, verifies

$$X(P) = OG = OE + EF + FG. \tag{1}$$

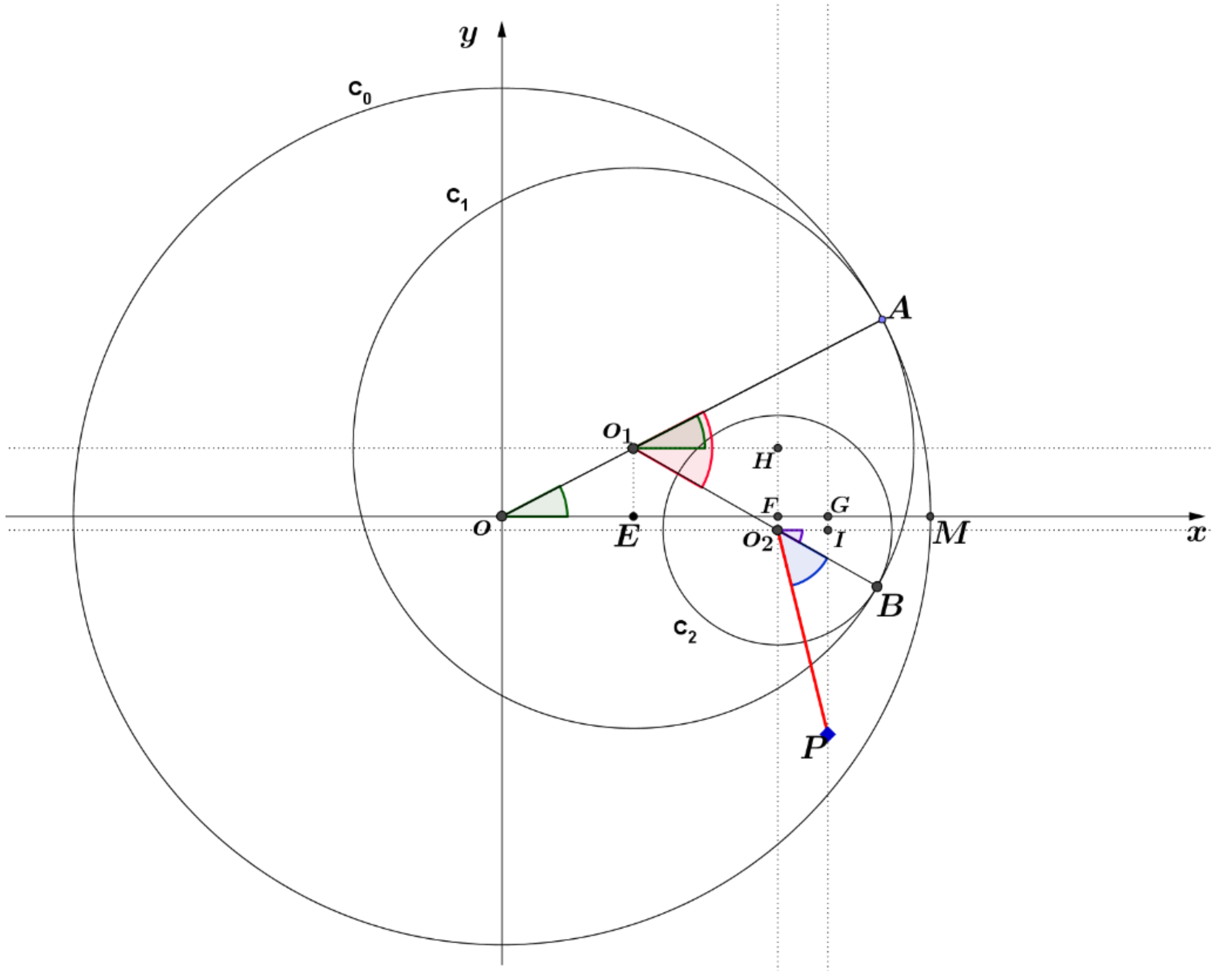


Figure 2: Three circles hypotrochoid
The elements of the proof of the calculation of the *X(P).*

In the follow, we calculate the length of each segment in the right side of Equation (1) in function of the anglese $\angle MOA$,$\angle AO_1B$ and $\angle BO_2P$. Indeed,

$$OE = \text{x-coordinate of } O_1 = OO_1 \cos(MOA) = (R - r_1) \cos(MOA). \tag{2}$$

Notice that $\angle HO_1A$ is equal to $\angle MOA$, being angle between paralels O_1H and edge Ox. Besides

$$\angle HO_1O_2 = \angle AO_1B - \angle AO_1H = \angle AO_1B - \angle MOA, \tag{3}$$

that, along with the trigonometric relations of the triangle O_2O_1H, nos fornece a conclusão:

$$EF = O_1H = O_1O_2 \cos(HO_1O_2) = (r_1 - r_2) \cos(AO_1B - MOA) \tag{4}$$

On the other hand, we have

$$\angle BO_2I = \angle O_2O_1H \tag{5}$$

In the right-angle triangle O_2IP, using the trigonometric relations and Equation (6), it is obtained

$$O_2I = O_2P\cos(PO_2I) = \text{d}\cos(AO_1B \quad - MOA + PO_2B\ ). \tag{7}$$

Consequently, the equality of the segments FG and O_2I is guaranteed. Therefore, using the Equations (1), (2) (4) and (7), we obtain that

$$X(P)= (R - r_1)\cos(MOA) + (r_1 - r_2)\cos(\text{A}O_1B - MOA) + d\cos(\text{A}O_1B - MOA + PO_2B) \tag{8}$$

From the dynamic proposed for this generalized hypotrochoid, we also have that

$$R.\ \text{arc}\ MOA = r_1.\text{arc}\ AO_1B = r_2.\ \text{arc}\ BO_2P,$$

Where arc MOA denoted the arc of a circle containing the points M and A. Denoting x = arc MOA, we conclude

$$\text{arc}\ AO_1B = x\ .R/r_1 \text{ and } \quad \text{arc}\ BO_2P, =x\ .R/r_2. \tag{9}$$

Substituting this last expression in Equation (8), we get

$$X(P) = (R - r_1)\cos(x) + (r_1 - r_2)\cos((R/r_1 - 1)\,x) + d\cos(\ (R/r_1 + R/r_2 - 1)\,x) \tag{10}$$

To calculate que the y-coordinate of P, and observing Figure 3, we have that que s = |y-coordinate P|,

$$\text{s} = OK + KL, \quad \text{and} \tag{11}$$

$$OK =|\ O_2 = JK - JO. \tag{12}$$

On the other hand, we have that $\angle O_1O_2S = \angle$BO2I, that together with Equations (3) and (5), yield

$$\angle O_1O_2S = \angle \text{BO}_2\text{I} = \angle O_2O_1H = \angle AO_1B - \angle MOA\ . \tag{13}$$

By the trigonometric relations in the right-angle triangle O_1SO_2, and using Equation (13), it is obtained

$$SO_1 = O_1O_2\sin(O_1O_2S) = (r_1 - r_2)\sin(AO_1B - MOA). \tag{14}$$

As we also have $SO_1 = JK$, then, by Equation (14), we conclude

$$JK = (r_1 - r_2)\sin(AO_1B - MOA). \tag{15}$$

Besides, By the trigonometric relations in the right-angle triangle OO_1E, we have

$$O_1E = OO_1\sin(MOA) = (R - r_1)\sin(MOA), \tag{16}$$

that together with the equality $O_1E = JO,$ and Equation (16) give us the expression

$$OJ = (R - r_1)\sin(MOA). \tag{17}$$

Observing in the right-angle triangle O_2QP we get the expression

$$O_2Q = O_2P\sin(QPO_2). \tag{18}$$

The equality of the angles $\angle IO_2P$ and $\angle O_2PQ$ in the rectangle O_2IPQ, along with Equations (6) and (18), inference that

$$KL = O_2Q = O_2P\sin(QPO_2) = O_2P\sin(IO_2P) = O_2P\sin(AO_1B - MOA + PO_2B) \tag{19}$$

Using Equations (11) and (12), we obtain that que o módulo daordenada de P é igual à medida de JK – JO + KL e, consequentemente, de (15), (17) and (19), verifies

$$X(P) = (r_1 - r_2)\sin(AO_1B - MOA) - (R - r_1)\sin(MOA) + O_2P\sin(AO_1B - MOA\ PO_2B). \tag{20}$$

Analogously, denoting x = arc MOA and using the equalities obtained in Equation (9), we can rewrite Equation (20) as follows:

$$s = (r_1 - r_2)\sin((R/r_1 - 1)x) - (R - r_1)\sin(x) + d.\sin((R/r_1 + R/r_2 - 1)x) \tag{21}$$

Observe that, from Figure 3, the y-coordinate of P, which we denote by Y(P) is negative, from where we conclude

$$Y(P) = (R - r_1)\sin(x) - (r_1 - r_2)\sin((R/r_1 - 1)x) - d.\sin((R/r_1 + R/r_2 - 1)x) \tag{22}$$

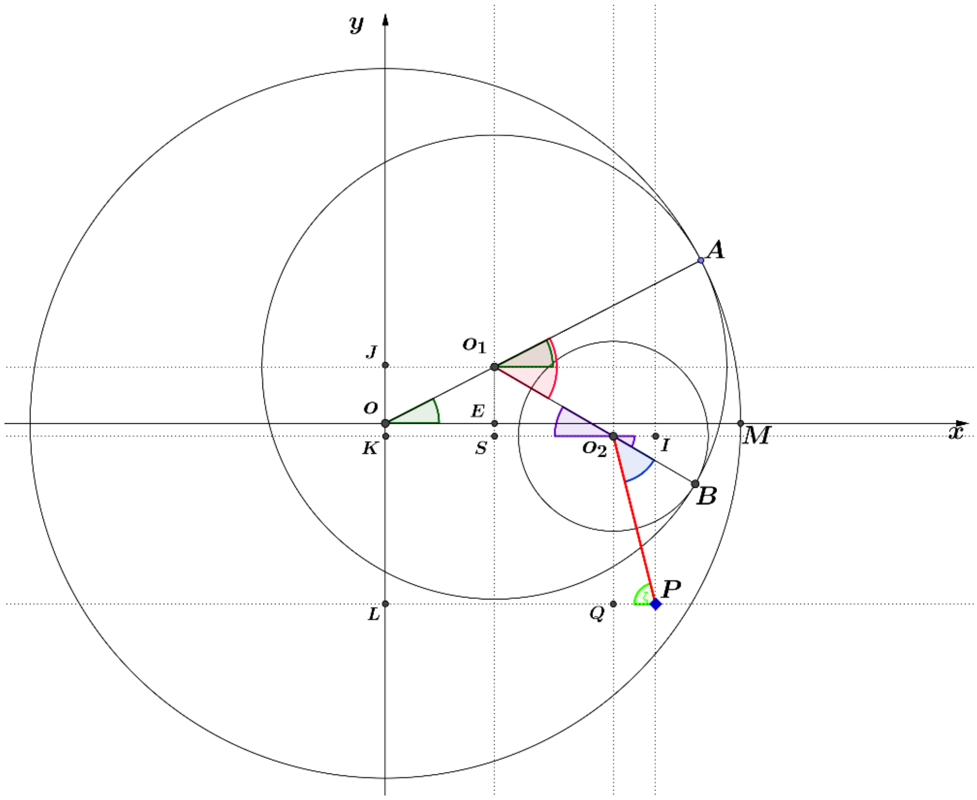


Figure 3: Three circles hypotrochoid

The elements of the proof of the calculation of the $X(P)$.

A similar proof could be done for the case of the epitrochoid that yields in the results of Equations (23) and (24):

$$X(P) = (R + r_1)\cos(x) + (r_1 + r_2)\cos((R/r_1 + 1)x) + d.\cos((R/r_1 + R/r_2 + 1)x), \tag{23}$$

$$Y(P) = (R + r_1)\sin(x) - (r_1 + r_2)\sin((R/r_1 + 1)x) - d.\sin((R/r_1 + R/r_2 + 1)x). \tag{24}$$

The formulas given by Equations (10), (22), (23) and (24) can be generalized for n circumferences, using the same dynamic by applying a mathematical induction method to obtain

$$\begin{aligned} X(P) = {} & (R - r_1)\cos(x) + (r_1 - r_2)\cos((R/r_1 - 1)x) \\ & + (r_2 - r_3)\cos((R/r_1 + R/r_2 - 1)x) + \ldots \\ & + (r_{n-1} - r_n)\cos((R/r_1 + R/r_2 + \ldots + R/r_{n-1} - 1)x) \\ & + d\cos((R/r_1 + R/r_2 + \ldots + R/r_n - 1)x), \end{aligned} \tag{25}$$

$$\begin{aligned} X(P) = {} & (R - r_1)\sin(x) - (r_1 - r_2)\sin((R/r_1 - 1)x) \\ & - (r_2 - r_3)\sin((R/r_1 + R/r_2 - 1)x) - \ldots \\ & - (r_{n-1} - r_n)\sin((R/r_1 + R/r_2 + \ldots + R/r_{n-1} - 1)x) - d\sin((R/r_1 + R/r_2 + \ldots + R/r_n - 1)x). \end{aligned} \tag{26}$$

Simulations built in the GeoGebra environment which curves follow the trajectory of the points with coordinator of Equations (10) and (22), are shown in Section 2 for n=2 and n=3.

3. First construction in GeoGebra

For the construction of the simulations in the GeoGebra environment, first is defined a slider T that represents the time control of the dynamic of the generalized hypotrochoid curve. The time T is measure in a way that the trajectory of the pole P designs a complete curve just once. In this construction, the slider T takes values between 0 and 2π. Equations (10) and (22) of the x-coordinator of P, are use in three stages:

1. Introduzing the functions

   $$a(x) = R\cos(x),$$

   $$b(x) = R\sin(x),$$

   to obtain the coordinators of the center O of the fixed circumference C0, substituting x for the value of the slider T, preciously defined.
2. In the input bar also are typed the following functions:

$$c(x)=(R - r_1)\cos(x) + (r_1 - r_2)\cos((R / r_1 - 1)\,x),$$

$$d(x) =(R - r_1)\sin(x) - (r_1 - r_2)\sin((R / r_1 - 1)\,x),$$

to obtain the coordinators of center O1 of the mobile circumference C1, substituting x for the value of the slider T.

3. The O_2 coordinators of the circumference center C_2, are obtained through the formulas

$$e(x)= (R - r_1)\cos(x) + (r_1 - r_2)\cos((R / r_1 - 1)\,x) + d\cos((R / r_1 + R / r_2 - 1)\,x)$$

$$f(x) = (R - r_1)\sin(x) - (r_1 - r_2)\sin((R / r_1 - 1)\,x) - d\sin((R / r_1 + R / r_2 - 1)\,x),$$

replacing x for the slider T values.

Therefore, determining the P coordinators as in Equation we get at instant T=0, for the corresponding values R = 20, $r_1 = 5$, $r_2 = 1$ and d = 5, the curve shown in Figure 4. When T = 2, the curve trajectory is shown in Figure 5.

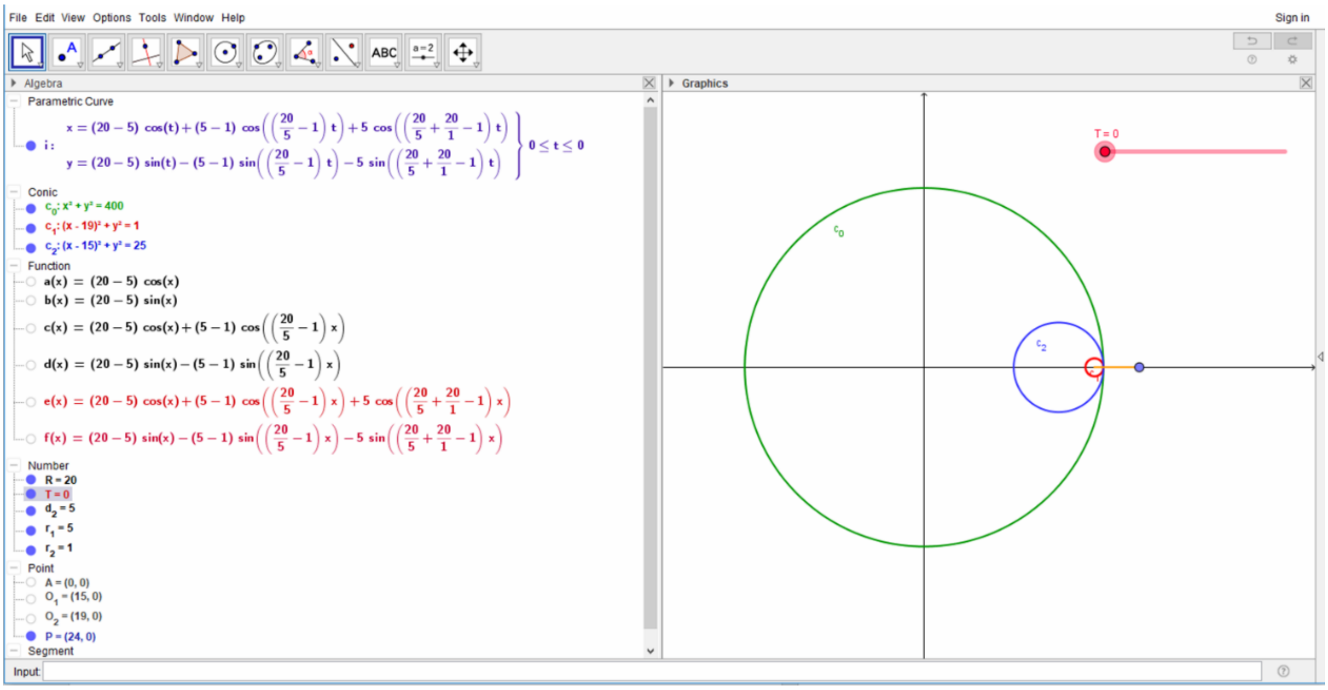


Figure 4: The generalized hypotrochoid curve of three circles.
First construction in GeoGebra. The slider T (time) at point 0.

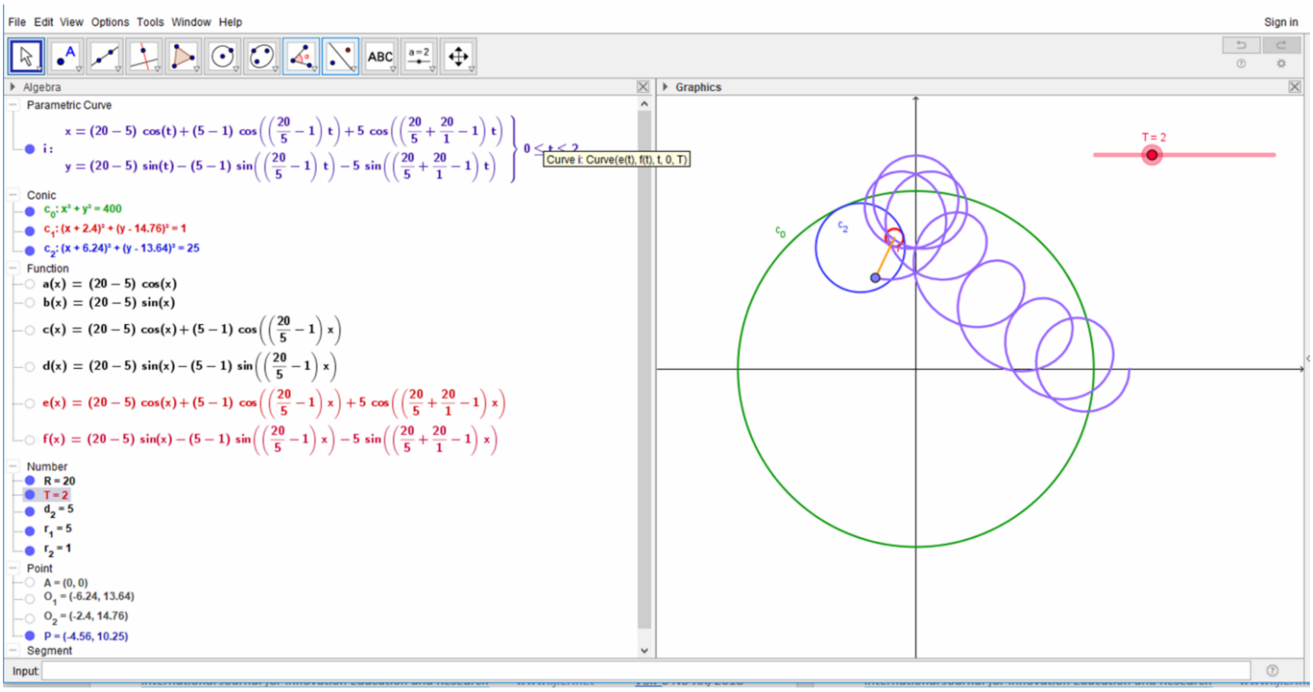


Figure 5: The generalized hypotrochoid curve of three circles.
The slider T in the same simulation of Figure 4 with T=2.

When T = 2π, the curve complete trajectory is shown in Figure 6.

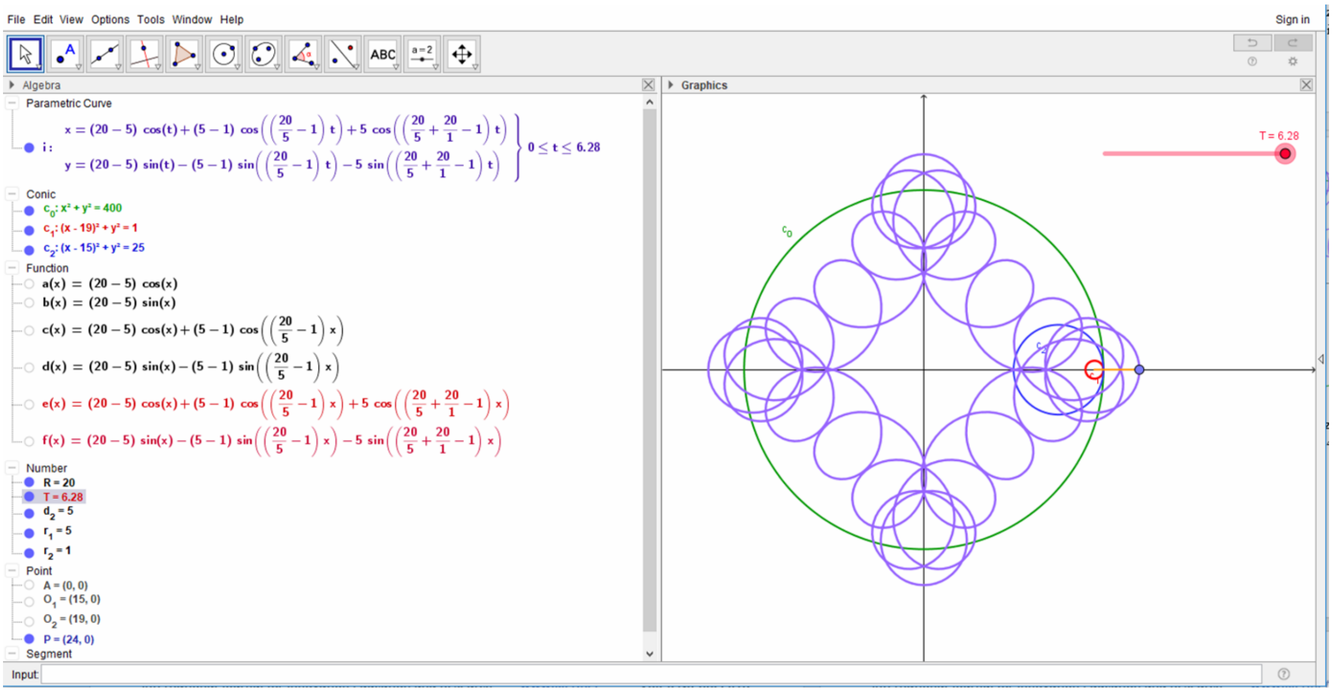


Figure 6: The generalized hypotrochoid curve of three circles.
The completion of the Figure 4 simulation of with the slider T at point 2π.

Many other simulations were performed with different values of R, r_1, r_2 and d. A more complex complete trajectory of the generalized hypotrochoid, is shown in Figure 7 the values R=20, r_1=8, r_2=1 and d=3 apply at T=0. In Figure 8, the same curve is shown at T=4π.

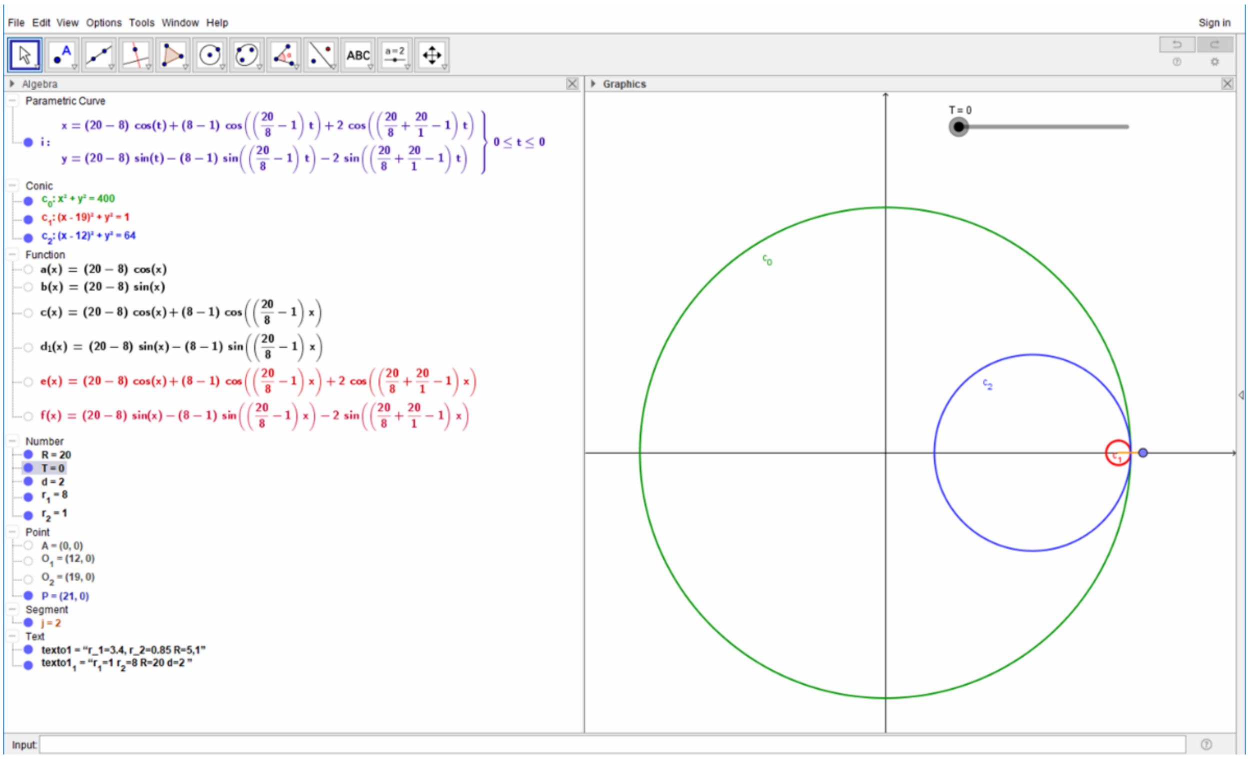
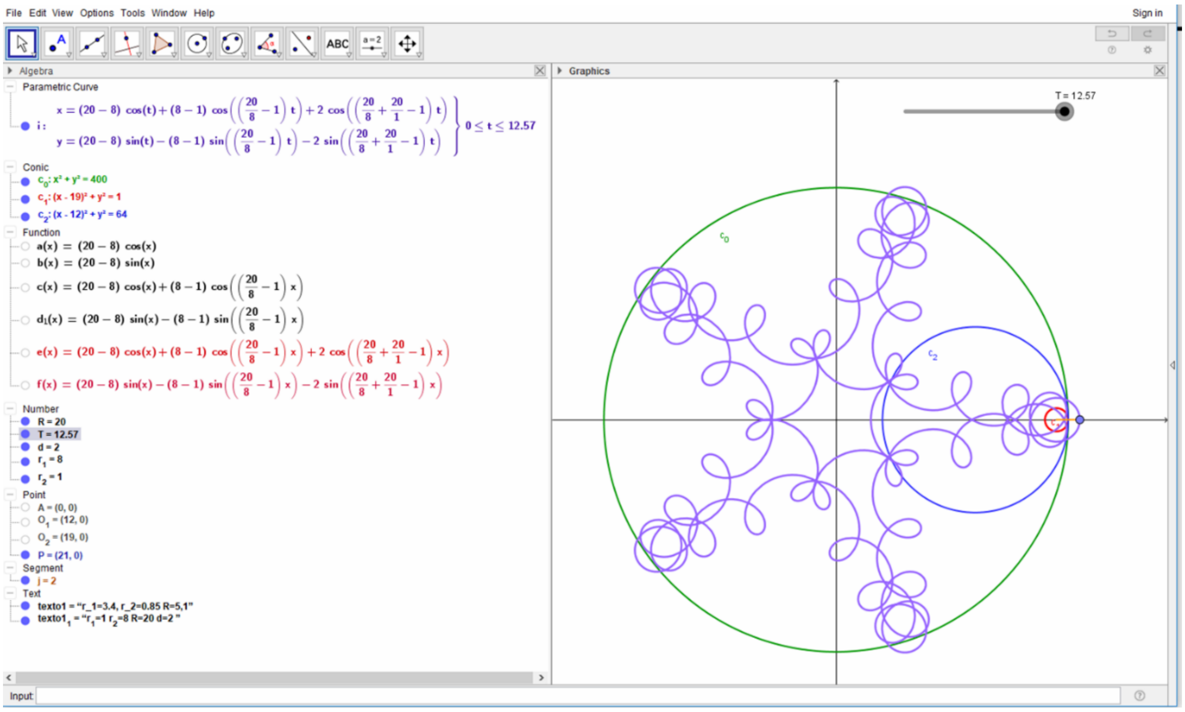


Figure 7: The generalized hypotrochoid curve of three circles.
Another construction in GeoGebra with the slider T at T=0.



0.

Figure 8: The generalized hypotrochoid curve of three circles.
The completion of the Figure 7 simulation with T=4π.

5. Conclusion

An explanatory research is carried out, trying to understand the cause of the dynamics of a classic curve, the hypotrochoid. It is also common to find explanatory research to support a descriptive research [4]. It is also sought to advance the results of the researches found, and rigorously demonstrate the findings.We seek to understand and describe the generalized mechanics of a curve. The task of finding a formula to mathematically base the discovery, is also facilitated by GeoGebra.The implementation and visualization in its environment were elements that contributed strongly to prove the theoretical results involved in obtaining the formulas.The time dynamics modeled with a slider, creates the fascinating images and the idealized movement being visualized through this tool.

It is important to note that, this study confirmed that this type of investigative process exceeded the expectations of students of scientific initiation, and that, according to them, has increasingly reinforced the importance of using GeoGebra as a facilitator for the enhancement of creativity through implementation of routines and visualization. And as a way of confirming rigorous mathematical demonstrations.

6. References

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