Applications of Continuous Fractions in Orthogonal Polynomials[1]

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Abstract

Several applications of continuous fractions are restricted to theoretical studies, such as problems associated with the approximation of functions, determination of rational and irrational numbers, applications in physics in determining the resistance of electric circuits and integral equations and in several other areas of mathematics. This work aimed to study the results that open the way for the connection of continuous fractions with the orthogonal polynomials. As support, we will study the general case, where the applications of the Wallis formulas in a monolithic orthogonal polynomial, which generates a continuous fraction of the Jacobi type. It will be allowed applications with relations of recurrence of three terms in the polynomials of Tchebyshev and Legendre, through the results found,

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establishing connection between them with the continuous fractions. And finally, will be presented the "Number of gold", that is an application of this theory.

Keywords: Jacobi, irrational numbers, infinite sequence.

1. INTRODUCTION

Continuous fractions were already studied and applied in several areas of knowledge since the 5th century, according to records of the time. The first known study on the properties of continuous fractions was made by Rafael Bombelli (1526-1572) in the edition of the book Algebra in 1573, shortly after his death [21], in which an approximation equal to that of irrational numbers was determined and the approximation of $\sqrt{13}$:

$$\sqrt{13} \simeq 3 + \frac{4}{6 + \frac{4}{6}} = \frac{18}{5} \tag{1}$$

which is known by the formula [1]:

$$\sqrt{a^2 + b} \simeq a + \frac{b}{2a} + \frac{b}{2a} + \frac{b}{2a} + \cdots \tag{2}$$

In the same way, Pietro Cataldi (1548-1626) [6] presented a development for $\sqrt{18}$:

$$\sqrt{18} \simeq 4 + \cfrac{2}{8 + \cfrac{2}{8 + \cfrac{2}{8 + \ddots}}} = 4 + \cfrac{2}{8 + \cfrac{2}{8 + \cfrac{2}{8 + \ddots}}} \tag{3}$$

abbreviated as follows:

$$4 + \frac{2}{8} + \frac{2}{8} + \frac{2}{8} \ldots \tag{4}$$

[6], states that it is in the time of 1692 that the continuous fractions are once again objects of study. Jhon Wallis (1616-1703) presented in his book Opera Mathematica, where it was based on the definitions of Lord Brouncker (1620 - 1684), the following definition:

$$\frac{4}{\pi} = \frac{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times 9 \ldots}{2 \times 4 \times 4 \times 6 \times 6 \times 8 \times 9 \ldots} \tag{5}$$

which consequently would later be a north for studies carried out by Euller (1707-1783), where it demonstrated the root of a quadratic polynomial in the form of continuous fraction, and contributes it to that expression of the number e [2].

According to the author, at the beginning of the twentieth century great advances were made in the theory of continuous fractions in relation to the studies carried out by Wallis and Euller, both in the use of computational algorithm, approximations of rational and irrational numbers, and in the resolution of diophantine equations [25]**Error! No bookmark name given.**.

Equally important, according to [18], orthogonal polynomials are used to solve various problems of Applied Analysis because they are part of a family of special functions.

Because they satisfy a recurrence relation of three terms, they create the possibility of connecting with several mathematical themes, especially with the continuous fractions, which by means of the monolithic

orthogonal polynomials and the formulas of Wallis make such a connection a reality.
Above all, it is worth mentioning that even though the development of both themes has taken several generations, it is a field that grows on a large scale.
The objective of the present work is the accomplishment of a literary revision on the continuous fractions allowing the critical reflection on concepts, theories, methods and techniques of approach.

2. MATERIALS AND METHOD

Bibliographical revisions were made on the historical aspects of continuous fractions, concomitantly with their properties, convergences and connections with the orthogonal polynomials in order to develop the relevant objectives in the present study. The elaboration of the literary revision consists in the construction through several sources of research of discussion between authors resulting in the final consideration of the work, which will enable in a critical reflection on concepts, theories, methods and techniques of approach [16].

2.1 Continuous Fractions

According to authors [12] and [7], a rational number is any and every number that can be written as a ratio p/q, where p it is a whole and q a natural $(q > 0)$.
Applying the Euclid algorithm successively in a ratio, we obtain the general form of the continuous fraction, according to [1]:

$$\frac{p}{q} = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \ddots + \cfrac{b_n}{a_{n-1} + \cfrac{b_n}{a_n}}}} \tag{6}$$

where, $a_0, a_1, a_2, \ldots$, are positive integers and $b_0, b_1, b_2, \ldots$, are real numbers. Another way of denotation of the continuous fraction (6) according to [11] is:

$$a_0 + \frac{b_1}{a_1} + \frac{b_2}{a_2} + \cdots + \frac{b_n}{a_n}, \tag{7}$$

the partial quotient being defined by b_i/a_i, with $i = 1, 2, \ldots$.
The convergence of a continuous fraction from form (6) to a number p/q means the convergence of the convergent sequence $\{C_n\}$ to p/q and, n which case the continuous fraction value is $\{C_n\}_{n=0}^{\infty}$ of the sequence, which is written as follows [17]:

$$C_n = a_0 + \frac{b_1}{a_1} + \frac{b_2}{a_2} + \cdots + \frac{b_n}{a_n} \tag{8}$$

in which C_n represents the finite continuous fraction, called the nth term converging from fraction (7) and to which fraction (7) converges to K, from the following concept.

Definition 1. According to [1] we say that (7) being a continuous fraction, it converges to K (finite) if there exists at most a finite number of C_n, indefinite and $\lim_{n\to\infty} C_n = K$. If it does not occur, we affirm that the continuous fraction is divergent.

According to [18] of relation (8) one can write in the form $C_n = p_n/q_n$, with $n = 0, 1, 2, \ldots$, where:

$$\begin{array}{ll} p_0 = a_0 & q_0 = 1 \\ p_1 = a_0a_1 + b_1 & q_1 = a_1 \\ p_2 = a_0a_1a_2 + a_0b_2 + b_1a_2 & q_2 = a_1a_2 + b_2 \\ \vdots & \vdots \end{array} \quad (9)$$

where (9) are known as the Wallis formula and it is worth remembering that $p_2 = a_2p_1 + b_2p_0$ and $q_2 = a_2q_1 + b_2q_0$. In this way the following result is obtained using the principle of finite induction [3].

Theorem 1. According to [6], consider the sequences given by $\{p_n\}$ and $\{q_n\}$ where:

$$p_n = a_np_{n-1} + b_np_{n-2} \qquad q_n = a_nq_{n-1} + b_nq_{n-2} \quad (10)$$

with $p_{-1} = 1$, $p_0 = a_0$, $q_0 = 1$ and $a_n \neq 0$ para $n \geq 1$. Then, the nth convergent C_n satisfies:

$$C_n = \frac{p_n}{q_n}, \; n = 0, 1, 2, \ldots, \quad (11)$$

2.2 Orthogonal Polynomials

Orthogonal polynomials are used as solutions of problems in several applications of Applied Sciences. One of the applications of this extremely important theme is related to monic polynomials and the three terms. Thus, to determine the Jacobi polynomial a sequence of orthogonal polynomials associated with the function defined as weight [15] will be used.

Definition 2. According to [14], we consider (a, b) real interval defined by, $-\infty \leq a < b \leq \infty$, and the real function $\phi(x)$, so as to have infinite points of increase in (a, b), limited and non-decreasing. If $d\phi(x) = w(x)dx$, then $w(x) \geq 0$ in (a, b), will be called the weight function, provided that it is not identically zero.

Definition 3. A sequence $\{P_n(x)\}_{n=0}^{\infty}$ will a sequence of orthogonal polynomials with respect to the function $w(x)$, weight, in interval (a, b) if: " [14]:

(i) $P_n(x)$ is exactly degree $n \geq 0$.

(ii) $\langle P_n, P_m\rangle = \int_a^b P_n(x)\, P_m(x) w(x)dx, = \begin{cases} 0, & \text{if} \quad n \neq m, \\ p_n \neq 0 & \text{if} \quad n = m \end{cases}$ (12)

If $a_{n,n} = 1$, then the polynomials are monic, with denotation of $\{\hat{P}_n(x)\}_{n=0}^{\infty}$.

Since there is a recurrence of three terms by the orthogonal polynomials, the theorem follows:

Theorem 2. Consider $\{P_n(x)\}_{n=0}^{\infty}$ where (a, b), is a sequence of orthogonal polynomials relative to the weight function $w(x)$. Then, the following three-term recurrence relation must satisfy all the polynomials of this sequence [2]:

$$P_{n+1}(x) = (\gamma_{n+1}x - \beta_{n+1})P_n(x) - \alpha_{n+1}P_{n-1}(x), \qquad n \geq 0 \tag{13}$$

with $P_0(x) = 1$, $P_{-1}(x) = 0$, $\alpha_{n+1}, \beta_n;\ \gamma_n \in \mathbb{R}, n \geq 1$, and

$$\gamma_{n+1} = \frac{a_{n+1,n-1}}{a_{n,n}}, \beta_{n+1} = \gamma_{n+1}\frac{\langle xP_n, P_n\rangle}{\langle P_n, P_n\rangle} \text{ and } \alpha_{n+1} = \frac{\gamma_{n+1}}{\gamma_n}\frac{\langle P_n, P_n\rangle}{\langle P_{n-1}, P_{n-1}\rangle} \tag{14}$$

Recurrence for the monic polynomials is described in [23]:

$$\hat{P}_{n+1}(x) = \left(x - \hat{\beta}_{n+1}\right)\hat{P}_n(x) - \hat{\alpha}_{n+1}\hat{P}_{n-1}(x), \qquad n \geq 0 \tag{15}$$

with $\hat{P}_0(x) = 1, \hat{P}_{-1}(x) = 0$, and $\hat{\beta}_n = \gamma_{n+1}\frac{\langle x\hat{P}_{n-1}, \hat{P}_{n-1}\rangle}{\langle \hat{P}_{n-1}, \hat{P}_{n-1}\rangle}$ e $\alpha_{n+1} = \frac{\langle \hat{P}_n, \hat{P}_n\rangle}{\langle \hat{P}_{n-1}, \hat{P}_{n-1}\rangle}$.

2.3 Polynomials associated with orthogonal polynomials

Orthogonal polynomials are related to the polynomials associated with three conditions. There are several applications related to its use, in which many are used in Applied Analyzes, involving problems of fundamental role in the sciences and engineering, in which we highlight the orthogonal polynomials associated to the known classic measures, as is the case of Jacobi, Hermite and Laguerre [2].

Definition 4. Given $\{\hat{P}_n(x)\}_{n=0}^{\infty}$, a sequence of monolithic orthogonal polynomials, we define the polynomial associated with $\hat{P}_n(x)$ by [2]:

$$Q_n(x) = \int_a^b \frac{\hat{P}_n(t) - \hat{P}_n(x)}{t - x} w(t)dt, \qquad n \geq 0 \tag{16}$$

According to the author, it can be shown that $Q_n(x)$ is defined as a polynomial degree of $n - 1$; where μ_0 is the degree of the largest term of the coefficients. By the above definition and due to the existence of the recurrence relationship, the associated polynomials $Q_n(x)$ correspond to the same relation of the orthogonal polynomials [23].

2.4 Jacobi Polynomials

According to [15], the Jacobi polynomials are defined by $P_n^{(\alpha,\beta)}$ and in the interval $[-1,1]$ have definitions in association with the weight function:

$$w(x) = (1 - x)^{\alpha}(1 + x)^{\beta}, \qquad \alpha > -1, \qquad \beta > -1 \tag{17}$$

Also according to author, these polynomials are given by the formula:

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n\Gamma(n + \alpha + \beta)}{\Gamma(2n + \alpha + \beta)}(1 - x)^{-\alpha}(1 + x)^{-\beta}\frac{d^n}{dx^n}\left[(1 - x)^{n+\alpha}(1 + x)^{n+\beta}\right] \tag{18}$$

in monistic form, we have:

$$\Gamma(t) = \int_o^{\infty} e^{-x}x^{t-1}\, dx, \qquad t \in \mathbb{C}\ e\ Re(t) > 0 \tag{19}$$

known as Gamma function.

Jacobi's polynomials can be developed by the three-term recurrence relation [2]:

$$P_{n+1}^{(\alpha,\beta)}(x) = \left(x - \beta_{n+1}^{(\alpha,\beta)}\right) P_n^{(\alpha,\beta)}(x) - \alpha_{n+1}^{(\alpha,\beta)} P_{n+1}^{(\alpha,\beta)}(x), \qquad n > 1 \tag{20}$$

on what:

$$\alpha_{n+1}^{(\alpha,\beta)} = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}$$

$$\beta_{n+1}^{(\alpha,\beta)} = \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta+2)(2n+\alpha+\beta)},$$

$$P_o^{(\alpha,\beta)}(x) = 1, P_1^{(\alpha,\beta)}(x) = x + \frac{\alpha-\beta}{\alpha+\beta+2}$$

3. RESULTS AND DISCUSSIONS

3.1 General case

Using the formulas of Wallis (10)] in the general form of the orthogonal polynomials, we have according to [5]:

$$C_n = a_0 + \frac{b_1}{a_1} + \frac{b_2}{a_2} + \cdots + \frac{b_n}{a_n} \Longrightarrow C_n(x) = \frac{p_n}{q_n} = \frac{(x-\hat{\beta}_n)p_{n-1}(x) - \hat{\alpha}_n p_{n-2}(x)}{(x-\hat{\beta}_n)q_{n-1}(x) - \hat{\alpha}_n q_{n-2}(x)} \tag{21}$$

where $\hat{\alpha}_1 = \mu_0,\ p_{-1}(x) = 1,\ p_0(x) = 0,\ q_{-1}(x) = 0$ and $q_0(x) = 1$. In this way, we obtain:

$$C_n = \frac{\mu_0}{x-\hat{\beta}_1} - \frac{\hat{\alpha}_2}{x-\hat{\beta}_2} - \frac{\hat{\alpha}_3}{x-\hat{\beta}_3} - \cdots - \frac{\hat{\alpha}_n}{x-\hat{\beta}_n} \tag{22}$$

with $\hat{\beta}_n$ and $\hat{\alpha}_{n+1},\ n \geq 1$, are coefficients of the recurrence relation for the orthogonal polynomials $\hat{P}_n(x)$, where this polynomial is related to a weight function $w(x)$ [5].

It is verified that the recurrence relation of $q_n(x)$ is the relation defined for the polynomials associated with the orthogonal polynomial $Q_n(x),\ n \geq 2$. Analyzing that $q_0(x) = Q_n(x) = 0$ and $q_1(x) = Q_1(x) = \mu_0$, we see that $q_0(x) = Q_n(x), n \geq 0$ [2].

Therefore $P_n(x) = \hat{P}_n(x),\ n \geq -1$, with the nth of the continuous fraction (18) given by:

$$C_n = \frac{\mu_0}{x-\hat{\beta}_1} - \frac{\hat{\alpha}_2}{x-\hat{\beta}_2} - \frac{\hat{\alpha}_3}{x-\hat{\beta}_3} - \cdots - \frac{\hat{\alpha}_n}{x-\hat{\beta}_n}, \quad n \geq 0 \tag{23}$$

(18) is defined as the continuous fraction of Jacobi or J-fraction.

It is worth mentioning the existence of a sequence of orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$ associated with a weight function $w(x)$, where they satisfy the three-term recurrence relationship, thus satisfying the polynomials orthogonals [23].

It is considered the polynomials of Tchebyshev and Legendre, other applications of the continuous fractions in orthogonal polynomials with respect to the recurrence of three terms.

3.2 Polynomials of Tchebyshev

The Tchebyshev polynomials of second species $Un(x)$ are defined as follows [2]:

$$U_n(x)\frac{sin[(n+1)\,arccos\,x]}{\sqrt{1-x^2}}, \qquad x \in [-1,1], \qquad n = 0,1,2,\ldots \tag{24}$$

in this way:

$$U_0(x) = 1$$

$$U_1(x) = 2x$$
$$U_2(x) = 2^2x^2 - 1$$
$$U_3(x) = 2^3x^3 - 4x$$

Still according to author, it is necessary to determine two properties [2]:

a) The recurrence relation of three terms:

Using Trigonometric Formulas and Relationships.

$$\sin[(n+2)\theta] + \sin(n\theta) = 2\cos\theta.\sin[(n+1)\theta] \tag{25}$$

being the value of $x = \cos\theta$ we have the following:

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \tag{26}$$

where it will be valid for $n \geq 0$ if we use $U_{-1}(x) = 0$.

b) Those are orthogonal in the interval $[-1, 1]$ with respect to the weight function $w(x) = \sqrt{1-x^2}$.

With the development of the integral defined as follows:

$$\int_0^{\pi} \sin(k\theta)\sin(j\theta)d\theta = 0 \tag{27}$$

For values of k and j integers, and $k \neq j$, we have:

$$\int_0^{\pi} \frac{\sin[(n+1)\theta]}{\sin\theta}\frac{\sin[(m+1)\theta]}{\sin\theta}\sin^2\theta d\theta = \int_{-1}^{1} U_n(x)\,U_m(x)\sqrt{1-x^2}dx = 0 \tag{28}$$

for $m \neq n$. With $m = n$ the set value of the integral will be equal to $\pi/2$.

The recurrence relation allows the natural union of the continuous fractions with the polynomials, where $U_n(x)$ is nth convergent of the continuous fraction [1]:

$$\frac{-1}{2x} + \frac{-1}{2x} + \frac{-1}{2x} + \cdots \frac{-1}{2x} + \cdots \tag{29}$$

The polynomial $U_n(x)$ has degree n and its coefficient with greater degree is 2^{-n}. Therefore, the monoclonal polynomial of Tchebyshev is its described in the form:

$$\widehat{U}(x) = 2^{-n}U_n(x), \qquad n \geq 0 \tag{30}$$

As soon as,

$$\widehat{U_0}(x) = 1, \qquad \widehat{U_1}(x) = x \tag{31}$$

Case $\widehat{U}_{-1}(x) = 0$ in general we have:

$$\widehat{U}_{n+1}(x) = x\widehat{U}_n(x) - \frac{1}{4}\widehat{U}_{n-1}(x), \qquad n \geq 0 \tag{32}$$

All in all, the classical Tchebyshev polynomials are considered to be denominators of the converged ones of the continuous fraction [1].

$$\frac{-\frac{1}{2}}{x} + \frac{-\frac{1}{4}}{x} + \frac{-\frac{1}{4}}{x} + \cdots + \frac{-\frac{1}{4}}{x} + \cdots \tag{33}$$

Even so, we have valid orthogonality, because:

$$\int_{-1}^{1} \widehat{U}_n(x)\widehat{U}_m(x)\sqrt{1-x^2}dx = \frac{\pi}{2}2^{-(m+n)}\delta_{mn} \tag{34}$$

where δ_{mn} is the Kronecker delta with the value 0 if $m \neq n$ and 1 if $m = n$.

3.3 Legendre Polynomials

Legendre's equations have the property of defining the first solution of the equation in which it bears the same name, Legendre's equation [3]. Thus, according to [3] and [8], the Legendre equation has many of its applications in the so-called differential equations in interest mainly in the physical-mathematical area, usually applied as an introductory mode of special functions, in which the expansion in series of powers, known as the Frobenius Method.

Another form of connection with the continuous fractions is the Legendre polynomial, where $Pn(x)$ is given by $P_0(x) = 1$ and $P_1(x) = x$, and in general [3]:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \tag{35}$$

considering $n \geq 0$, when the value of $P_{n-1}(x) = 0$, we find the first polynomials:

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$$

$$P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$$

Legendre's polynomials are orthogonal in the interval $[-1,1]$ with respect to the weight function $w(x) = 1$. So,

$$\int_1^1 P_m(x)P_n(x)dx = \frac{n}{n+1}\delta_{mn} \tag{36}$$

Thus the recurrence relation of three terms will be written as follows:

$$P_{n+1}(x) = \frac{2n+1}{n+1}xP_n(x) - \frac{n}{n+1}P_{n-1}(x) \tag{37}$$

If the value for $a_0 = 0$, $b_0 \neq 0$, in the continuous fraction (7), with value of $b_{n+1} = -\frac{n}{n+1}$ and $a_n = \frac{2n-1}{n}x$ for values of $n \geq 1$, we obtain:

$$\frac{b_1}{x} + \frac{-\frac{1}{2}}{\frac{3}{2}x} + \frac{-\frac{2}{3}}{\frac{5}{3}x} + \frac{-\frac{3}{4}}{\frac{7}{4}x} + \cdots + \frac{-\frac{n}{n+1}}{\frac{2n+1}{n+1}x} + \cdots \tag{38}$$

According to the author [1], the recurrence ratio of the continuous fraction (38) with numerators $p_n = p_n(x)$ and denominators $q_n = q_n(x)$ of the convergent $C_n = C_n(x)$ being $n \geq 1$ is defined in (37). For $q_0(x) = P_0(x) = 1$ and $q_1 = P_1(x) = x$, the sequence $P_n(x)\}_{n=1}^{\infty}$ is considered the sequence of the denominators of the convergent ones of the continuous fraction (38).

According to the same authors, if the numerator and denominator of a fraction are multiplied by a factor other than zero, the value of the fraction would not change. Thus, by using the term $2/3$ for the numerator and denominator of the second fraction, $5/3$ for the third, and so on, will result in the continuous fraction, equivalent to (38).

$$\frac{b_1}{x} + \frac{-\frac{1^2}{1.3}}{x} + \frac{-\frac{2^2}{3.5}}{x} + \cdots \tag{39}$$

Thus, the recurrence relation for the denominators is defined by $\hat{P}_{n+1}(x) = x\hat{P}_n(x) - \frac{n^2}{4n^2-1}\hat{P}_{n-1}(x)$, and with initial values $\hat{P}_0(x) = 1$ and $\hat{P}_1(x) = x$, we find:

$$\hat{P}_2(x) = x^2 - \frac{1}{3}$$
$$\hat{P}_3(x) = x^3 - \frac{5}{3}x$$
$$\hat{P}_4(x) = x^4 - \frac{202}{105}x^2 + \frac{3}{35}$$

Thus, the polynomial $\hat{P}_n(x) = x^n + \cdots$ is constituted by the sum of x^n with terms that have exponents smaller than n in x, that is, $\hat{P}_n(x)$ is considered a monic polynomial with orthogonality . So we have:

$$\int_{-1}^{1} \hat{P}_m(x)\hat{P}_n(x)dx = 0, \qquad m \neq n \tag{40}$$

3.4 Golden Ratio

The golden ratio is characterized by the ratio of the sum of two distinct segments to the shortest segment, approaching the result of the irrational number *phi* ϕ, of which the value is approximately $1{,}618$ ... []. The rectangle is a figure marked by the harmony and regularity of its features and measures having a great power of influence in history, becoming a pillar for the works of art [22]. According to the authors [4] and [9], the golden rectangle consists of the relationship between its base and height in the irrational number *phi* ϕ, whose formation of the rectangle occurs by the way of a square in which a transverse from the vertex of that square formed with the middle point of its base, so as to form an arc with center in it, thus giving rise to the golden rectangle (Figure 1).

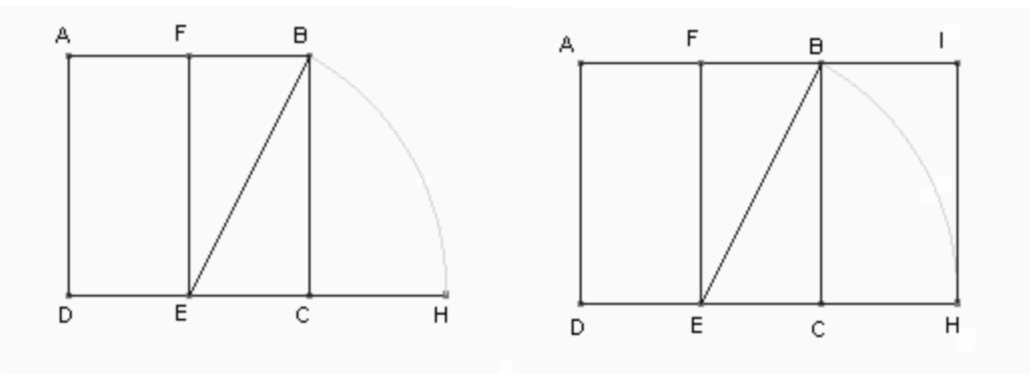

Figure 1: Golden Retangle [4]

We also observe that same number present in the relation of the Fibonacci sequence, whose value is established when dividing a term of the sequence by its predecessor, thus characterizing the number of gold, that thus, when higher the values of the sequence, the greater the accuracy in the determination of this

enigmatic irrational number, called the number of gold [13] and [10].
Let's see how the first nine terms of the Fibonacci sequence are shown [13]. Defining the sequence for any natural n [19]:

$$\begin{gathered} u(1) = 1, \qquad u(2) = 1 \\ u(n+1) = u(n-1) + u(n) \end{gathered} \tag{41}$$

So we have to,

$$\begin{gathered} f(n) = \frac{v(n+1)}{v(n)} = \frac{1}{1} = 1;\ \frac{2}{1} = 2;\ \frac{3}{2} = 1{,}5;\ \frac{5}{3} = 1{,}66 \ldots.; \\ \frac{8}{5} = 1{,}6;\ \ \frac{13}{8} = 1{,}625;\ \frac{21}{13} = 1{,}615 \ldots;\ \ \frac{34}{21} = 1{,}618 \ldots. \end{gathered} \tag{42}$$

It is observed at the limit below the sequence with the value of n tending to infinity, its limit is exactly the number ϕ *phi* [19].

$$\lim_{n\to\infty} \frac{f(n)}{f(n-1)} = \phi = 1{,}618 \ldots \tag{43}$$

Its application is in the most diverse areas, as in the arts expressed through the golden rectangles present in the works of Leonardo da Vinci, as well as in the architecture of Greek constructions, in addition to its presence in nature with surprising proportions and relations; whether with of the human body, of the ratio between total height by the measured segment of the feet to the navel, as well as among other relations of the body itself [20]; and is also found in nature itself, either in a simple flower with relation of the proportions of the floral petals of a jasmine, as well as in the construction of the shell of the Marine Nautilies in which the presence of the enigmatic number is noticed [20] [13].

4. CONCLUSION

Continuous fractions, as described in the paper, present vast applications in several areas of pure and applied mathematics. In this way, the present work presented the connections between the continuous fractions and the orthogonal polynomials, being developed by means of the mononomic polynomials for the general case and the relation of recurrence of three terms for the other cases demonstrated. The continuous fractions are summed up in a successive application of the Euclidean algorithm, so that its convergence for a given K(finite) value is defined in $\lim\limits_{n\to\infty} C_n = K$ finite number of C_n is undefined. Not getting the requirements described, the number divergent.

This convergence allows the connection with the orthogonal polynomials with a weight function, by way of the successive application of the Wallis formula in the monoclonal orthogonal polynomials, along with the relation of three terms, thus satisfying the connection.

Although not elaborated in detail, it can be observed that the development and concepts of the continuous fractions can complement the education of the pupils of the basic education, since its application occurs in irrational numbers, defined as part of the base of studies by the secretariat of the education in the state of São Paulo in the curricular proposal.

It stands out as an application of the continuous fraction, the relation of the convergent golden proportion to the irrational number 1,618 ..., which is represented by the Greek letter ϕ *(phi)*. Applications are found in the main works of art by the most varied artists of humanity such as Leonardo da Vinci, as well as a present in the architecture of ancient Greece, using the rectangle with golden proportions. In nature we can highlight the case of the human body, the Nautical Marine, among others.

However, the present work discloses part of the theoretical concepts about continuous fractions applied in the orthogonal polynomials, and therefore, the use of continuous fractions can now be seen as an easy-to-use and extremely important application in the world of exact science.

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