Model Reduction Methods Applied to A Nonlinear Mechanical System

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Abstract

Modern structures of high flexibility are subject to physical or geometric nonlinearities, and reliable numerical modeling to predict their behavior is essential. The modeling of these systems can be given by the discretization of the problem using the Finite Element Method (FEM), however by using this methodology, it is a very robust model from the computational point of view, making the simulation process difficult. Using reduced models has been an excellent alternative to minimizing this problem. Most model reduction methods are restricted to linear problems, which motivated us to maximize the efficiency of these methods considering nonlinear problems. For better accuracy, in this study, adaptations and improvements are suggested in reduction methods such as the Enriched Modal Base (EMB), the System Equivalent Reduction Expansion Process (SEREP), QUASI-SEREP and the Iterated Improved Reduced System (IIRS). The stability of a system is discussed according to the calculation of the Lyapunov exponents and phase space. Numerical simulations showed that the reduced models presented a good performance, according to the commitment of quality and speed of responses (or time saving).

Keywords*: model reduction methods, Nonlinear mechanical system; finite element method; stability analysis; computational cost.*

INTRODUCTION

Modern engineering projects are constantly motivated by research, aiming to study vibration and noise attenuation. Vibration analysis in mechanical systems can provide design improvements. Predicting the behavior of the structure is important to provide users with more security and comfort, avoiding excessive deflections and catastrophic mechanical failures, [1] and [2].

Structures comprising innovative elements of high flexibility, high resistance to soaring temperatures and operating speeds corroborate the study of physical or geometric nonlinearities that effectively describe physical phenomena in a system. Nonlinearities can affect several components of the system, such as inertial terms, parameters describing elastic and inelastic forces, external excitations and boundary conditions [2], [3], and [4]. Dynamic behavior of nonlinear systems is unpredictable due to a wide range of responses and phenomena such as chaos. The stability of the system, associated with the characteristics of a solution subject to disturbances, must be considered. Thus, if perturbation does not affect the system response significantly, the system is stable; otherwise, it is unstable [5].

Chaotic behavior must be correctly identified. Therefore, diagnostic tools are essential and system invariants are good alternatives for this purpose [5], [6] and [7]. Attractor dimensions, phase space, Poincare maps and Lyapunov exponents are some examples of tools used to identify chaotic behavior.

Lyapunov exponents are commonly used in research as diagnostic tools to define possible directions of local instability in the dynamics of the system. Studies on the stability of dynamical systems are shown in [5], [6], [7], [8] and [9]. Moreover, regarding nonlinearity, it is worth noting the stability study of systems for several timedelayed values in order to confirm their influence on vibration attenuation [10].

When modelling mechanical structures, elastic elements are represented by an infinite number of degrees of freedom and analytical solutions are rare or even impossible. The use of numerical-computational tools provides precise approximations of the behavior of these systems. In structures of industrial interest, the finite element method performs better and is one of the most used [1], [2] and [11]. For better accuracy, modelling structures by FEM procedures requires models with a large number of degrees of freedom, which is costly. Matrices of high order can incur extortionate computational costs. To minimize this problem, model reduction methods are constantly developed and studied [11].

In the literature, many reduction methods have been discussed in-depth and applied to FEM models. Reduction model methods are generally referred to as static or dynamic. Static methods are typically based on Guyan reduction and variations of it. Dynamic methods are based on real and complex modal analysis and modal methods [12], [13], [14] and [15].

The Static method or Guyan method, [16] and [17], is considered the first reduction method and is still very widespread in structural mechanics. This method provides good approximations in rudimentary problems and of static nature. However, in dynamic problems, the representation is flawed, which leads to the improvement and development of better reduction methods.

It was suggested that the improved Reduced System in [19] should include the forces of inertia of the structures as pseudo-static forces in the transformation matrix. However, it is not very suitable for verifying the orthogonality of modes [20]. Iterative approaches such as the Iterative Improved Reduced System (IIRS) were developed [21], [22]. Subsequently, improvements were suggested in the transformation matrix in order to accelerate the convergence process in [23]. Due to accuracy and accessible formulations, the IIRS method has led to the development of more efficient methods: multi-level condensation schemes for non-damping structural systems, where the choice of retained gases is influenced by the energy level of each element; combinations with other methods of sub-structuring; and accelerated substructure reduction procedures [23], [24] and [25].

Robust reduction methods were developed from the IIRS formulation as in: structural damage diagnosis algorithms for incomplete modal data [26]. The damage detection problem is formulated as an optimization problem and the static responses of the reduced model are calculated by the flexibility matrix; and dynamic algebraic condensation [27], when global matrices are subdivided and their substructures and interface boundaries are defined in the algebraic perspective. In the latter, the reduced model is obtained by condensation of substructure stiffness, reduction of interface contours and condensation of the substructure inertia effect.

Some methods use modal matrices including eigenvectors arranged in a specific column format such as the System Equivalent Reduction Expansion Process (SEREP), suggested in [29] and [30]. This allows extensive use of this method in elastic multibody problems [13], [14], [15], [31], [32] and [33]. A similar approach was suggested to reduce mechanical mass-spring systems with nonlinear models using the Quasi-Guyan method [34], [35] and [36]. In some studies, model reduction is based on a set of Ritz vectors. The transformation matrix or set of base vectors is obtained from the truncation of some normal modes followed by residual enrichment vectors [1], [2], [11] and [15]. Recent research, such as [37], emphasizes the importance of the correct choice of modes to be retained for the reliability and efficiency of the reduced model.

For the reduction of more complex structure models, the subdivision of components is used. The most knowledgeable method is Component Mode Synthesis (CMS), where each component can be reduced and analyzed independently without significant loss of system properties [38].

In other cases, approaches inspired by linear reanalysis methods are effective in reducing some complex problems. Methods such as Combined Approaches (CA) originally developed for linear reanalysis, have proven to be effective in cases when constitutive matrices of the system are variable and where reduced models using a fixed base reduction matrix are not able to represent the dynamic behavior of the system [2].

In this paper, model reduction methods are implemented according to some adaptations and improvements for better accuracy and to accelerate the speed of response of reduced models in nonlinear systems. In this paper, the Lyapunov exponents for finite element discretized systems are calculated and the stability of the system is discussed. The results obtained by the reduced models are compared with those of the complete model to verify the quality of response and time saving provided by each reduction method.

MODEL REDUCTION METHODOLOGY

In the reduction process, the reduced model is obtained by approximation of the discrete variables of the global system by a small number of approximation functions by transformation matrices.

In the structural model, the equation of motion with multiple degrees of freedom localized non-linearity is written as:

$$[M]\{\ddot{x}(t)\}+[D]\{\dot{x}(t)\}+[K]\{x(t)\}=\{F(t)\} \tag{1}$$

where $[M]$, $[D]$, $[K]$, order $n\times n$, are the mass, damping and stiffness matrices, respectively; $\{x(t)\}$ and $\{F(t)\}$, order $n\times 1$, are the vector displacement and load vector varying time. Rayleigh's proportional damping is assumed, $[D]= 0.005*[\mathrm{K}]$.

The concept of model reduction is to find a low dimension subspace $[T]$, order $n \times R$, with $R << n$, to approximate the state vector x, i.e.:

$$\{x_n\} \approx [T]\{x_R\} \qquad (2)$$

By substituting Eq (2) in Eq (1) and multiplying Eq. (1) by the matrix $[T]$, a new equation is obtained:

$$[M_R]\{\ddot{x}_R(t)\} + [D_R]\{\dot{x}_R(t)\} + [K_R]\{x_R(t)\} = \{F_R(t)\} \qquad (3)$$

where

$[M_R] = [T]^T [M][T]$; $[D_R] = [T]^T [D][T]$; $[K_R] = [T]^T [K][T]$ and $\{F_R(t)\} = [T]^T \{F(t)\}$.

ENRICHED MODAL BASE (EMB)

An alternative to reducing a FEM model is to consider one part of dynamic analysis conventional by low frequencies. To construct the modal base, only the first modes are considered. Thus, the reduced equations are obtained using a basis set that includes only a truncated set of vibration modes.

The reduced base vector can be obtained by solving the eigenvalue problem:

$$(K_0 - \lambda_i M)\varphi_i = 0 \qquad (4.a)$$

$$\varphi_0 = [\varphi_1 \; \varphi_2 \; \dots \; \varphi_r] \quad (4.b) \qquad \Lambda_0 = diag(\lambda_1, \lambda_2, \dots, \lambda_r) \quad (4.c)$$

where $i = 1,\dots,n$.

In models with nonlinearities, this base may result in some errors due to the truncation of the database [1]. Indeed, basis T is enriched by introducing the following residues formed by the static displacements associated to external excitations [11]:

$$\Delta R = K_0^{-1} f(t) \qquad (5)$$

The enriched reduced base vector for the system is given as:

$$T = [\varphi_0 \; \Delta R] \qquad (6)$$

ITERATED IMPROVED REDUCED SYSTEM (IIRS)

The generalized system eigenvalue problem can be described in the substructured form in relation to the degrees of freedom retained and omitted [22] and [23]:

$$\begin{bmatrix} K_{mm} & K_{ms} \\ K_{sm} & K_{ss} \end{bmatrix} \begin{bmatrix} \Phi_{mm} \\ \Phi_{sm} \end{bmatrix} = \begin{bmatrix} M_{mm} & M_{ms} \\ M_{sm} & M_{ss} \end{bmatrix} \begin{bmatrix} \Phi_{mm} \\ \Phi_{sm} \end{bmatrix} \Lambda_{mm} \qquad (7)$$

where K and M, order $n \times n$, are symmetric stiffness and mass matrices; Φ consists of the mass-normalized eigenvectors and $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_n)$. Only the first r modes are included in the above equation. The subscripts “m” and “s” represent the master and slave DoFs, respectively; the sizes of master and slave DoFs are assumed to be m and s with $n = m + s$; and the eigenvalues are arranged in ascending order. From the second

set of the above equation, Φ_{sm} hence can be expressed as,

$$\Phi_{sm} = -K_{ss}^{-1} K_{sm} \Phi_{mm} + K_{ss}^{-1}\left(M_{sm} \Phi_{mm} \Lambda_{mm} + M_{ss} \Phi_{sm} \Lambda_{mm}\right) \qquad (8)$$

Where t is the transformation matrix between Φ_{mm} and Φ_{sm}, which takes the form:

$$t = t_{Guyan} + K_{ss}^{-1}\left(M_{sm} + M_{ss} t\right) \Phi_{mm} \Lambda_{mm} \Phi_{mm}^{-1} \qquad (9)$$

where t_{Guyan} represents the transformation matrix of the Guyan reduction technique.
Writing Eq. (9) as

$$t = t_{Guyan} + t_d \qquad (10)$$

$$t_d = K_{ss}^{-1}\left(M_{sm} + M_{ss} t\right) \Phi_{mm} \Lambda_{mm} \Phi_{mm}^{-1} \qquad (11)$$

and

$$T = \begin{bmatrix} I_{mm} \\ t \end{bmatrix} = \begin{bmatrix} I_{mm} \\ t_{Guyan} \end{bmatrix} + \begin{bmatrix} 0 \\ t_d \end{bmatrix} = T_{Guyan} + \begin{bmatrix} 0 \\ t_d \end{bmatrix} \qquad (12)$$

where I_{mm} is the unit matrix of size $m\times m$.

Substituting Eq. (12) for $K_R = T^T KT$ and $M_R = T^T MT$, and after algebraic manipulations a new relation is obtained:

$$\Phi_{mm}\Lambda_{mm}\Phi_{mm}^{-1} = M_d^{-1} K_{Guyan} \tag{13}$$

where,

$$M_d = M_{Guyan} + \left(M_{ms} + t_{Guyan}^T M_{ss}\right) t_d \tag{14}$$

by substituting in Eq. (9),

$$t = t_{Guyan} + K_{ss}^{-1}\left(M_{sm} + M_{ss} t\right) M_d^{-1} K_{Guyan} \tag{15}$$

This leads to an iterated scheme, in which the equations are

$$t_{(k)} = t_{Guyan} + K_{ss}^{-1}\left[M_{sm} + M_{ss} t_{(k-1)}\right]\left[M_{d(k-1)}\right]^{-1} K_{Guyan} \tag{16.a}$$

$$T_{(k)} = \begin{bmatrix} I \\ t_{(k)} \end{bmatrix}_{mm} \tag{16.b}$$

$$M_{d(k-1)} = M_{Guyan} + \left(M_{sm} + t_{Guyan}^T M_{ss}\right) t_{d(k-1)} \tag{17}$$

$$K_{R(k)} = \left[T_{(k)}\right]^T K T_{(k)} \tag{18.a}$$

$$M_{R(k)} = \left[T_{(k)}\right]^T M T_{(k)} \tag{18.b}$$

and

$$K_{R(k)} \Phi_{m(k)} = M_{R(k)} \Phi_{m(k)} \Lambda_{m(k)} \tag{19}$$

QUASI-GUYAN METHOD (Q-GUYAN)

For this method, assume that the system coordinate vector $\{x_n\}$ is partitioned into master and slave sub-vectors:

$$\{x_n\} = \begin{Bmatrix} x_s \\ x_m \end{Bmatrix} \tag{20}$$

where $\{x_m\}$ is a master vector; $\{x_s\}$ is a slave vector; and $s + m = n$.

Given that m refers to the lowest undamped frequency modes obtained by solving the linear system eigenvalue problem without damping and zero forces, the associated modal matrix is partitioned as follows:

$$[\varphi]_{n\times m} = \begin{bmatrix} [\varphi_s] \\ [\varphi_m] \end{bmatrix} \tag{21}$$

where the $[\varphi_s]_{s\times m}$ is associated with the slave DoFs, and $[\varphi_m]_{m\times m}$ is associated with the master DoFs. Based on these matrices, a relationship matrix between the master and slave coordinates can be obtained in the following form,

$\{x_s\} = [R]_{s\times m}\{x_m\}$ (22) where $[R] = [\varphi_s][\varphi_m]^{-1}$. Thus, the reduction matrix $[T]$ is then defined so that

$$\{x\} = [T]\{x_m\};\ [T] = \begin{bmatrix} [R] \\ [I_m] \end{bmatrix} \tag{23}$$

For a thorough representation of nonlinearities in the model, improvements are proposed. Pre-multiplying Eq. (23) by the new matrix leads to

$$[T_0] = [S_0]\begin{bmatrix} [R] \\ [I_m] \end{bmatrix} \tag{24}$$

where $[S_0] = [K]^{-1}$, and considering the enrichment of the proposed basic vector in Eq.

(5), a new transformation matrix for this method is obtained

$$[T] = [T_0\ \Delta R] \tag{25}$$

2.4 SYSTEM EQUIVALENT REDUCTION EXPANSION PROCESS (SEREP)

In the formulation for this reduction strategy assuming that m is the number of dominant modes (generally, the modes lowest frequencies), determined by solving the eigenvalue problem of the version linear, without damping and unforced, the truncated solution of the eigenvalue problem may be expressed as:

$$\{x_n\} = [\varphi]\{p\} \tag{26}$$

where $[\varphi]_{n\times m}$ is the eigenvector matrix and $\{p\}_{m\times 1}$ is the modal participation vector.

In the second stage of reduction, all the states of the system are classified into two categories: the active (or master) states and omitted (or slave) states. There is no criterion for selecting the active states, however the number of active states is $\leq n$. Let a be the number of active states of a system and $d = n - a$ be the number of omitted states. Based on the division into master and slave sub-states of the system, Eq. (26) can be rewritten as

$$\begin{Bmatrix} x_a \\ x_d \end{Bmatrix} = \begin{bmatrix} [\varphi_a]_{a\times m} \\ [\varphi_d]_{d\times m} \end{bmatrix} \{p\}_{m\times 1} \tag{27}$$

There are now two different conditions depending on the number 'm' of the retained modes and the number 'a' of the active states: (1) $a \geq m$, which is the case commonly obtained in the structural dynamic problems and (2) $a < m$, which is a rare case but not impossible.

For case (1), $a \geq m$, and according to the first part of Eq. (27), the modal participation vector can be expressed in terms of the master DoF coordinates,

$$\{p\} = [\varphi_a]^{+}_{m\times a}\{x_a\} = \left[[\varphi_a]^T_{m\times a}[\varphi_a]_{a\times m}\right]^{-1}[\varphi_a]^T_{m\times a}\{x_a\} \tag{28}$$

where the subscript + signifies the pseudo-inverse. By substituting the relationship of Eq. (28) in Eq. (27), the coordinate transformation matrix T is defined,

$$\begin{Bmatrix} x_a \\ x_d \end{Bmatrix} = \begin{bmatrix} [\varphi_a]_{a\times m} \\ [\varphi_d]_{d\times m} \end{bmatrix} [\varphi_a]^{+}_{m\times a}\{x_a\} = [T]\{x_a\} \tag{29}$$

For this model reduction strategy, an improvement is proposed, similar to the previous case. By pre-multiplying Eq. (29) by the other matrix and considering the enrichment of the proposed basic vector in Eq. (5), a new transformation matrix for this method is obtained,

$$[T_{new}] = \left[[S][T] \;\; \Delta R\right] \tag{30}$$

where $[S] = -[K]^{-1}[M]$.

STABILITY OF THE NONLINEAR SYSTEM: LYAPUNOV EXPONENTS

Chaotic behavior must be correctly identified in dynamic systems. For this purpose, the calculation of Lyapunov exponents has been used as the most useful diagnostic tool for stability analysis of systems. The existence of positive exponents defines the local instability directions in the dynamics of the system.

Responses with at least one positive exponent present chaos, and responses with more than one positive exponent present hyperchaos [5], [6], [7], [8] and [9].

According to [5], given a sphere of states that is transformed by the dynamics of the system into an ellipsoid, the Lyapunov exponents are related to the nature of expansion and contraction of different directions in phase space. Thus, the relation between the initial sphere and the ellipsoid is considered in the evaluation of the divergence of two orbits. This variation can be expressed by:

$$d(t) = d_0 b^{\lambda t} \tag{31}$$

where d_0 is the initial diameter, b is a reference basis and λ is a Lyapunov exponent.

Hence, the Lyapunov spectrum is given by:

$$\lambda = \frac{1}{t}\log_b\left[\frac{d(t)}{d_0}\right] \tag{32}$$

In Eq. (32), when the Lyapunov exponent is negative or vanishes, trajectories do not diverge and when the exponent is positive it indicates that trajectories diverge, presenting chaos. Based on their values, dissipative systems have a negative sum of the whole Lyapunov spectrum.

In the literature, different algorithms to determine the Lyapunov exponent of time series are proposed. These algorithms are divided into two classes [5]: Trajectories, real space or direct method; and perturbation, tangent space or the Jacobian matrix method [6], [8] and [9].

Due to local exponential divergences of near orbits in chaotic situations, in the evaluation of the Lyapunov exponents one uses appropriate algorithms that evaluate the average of this divergence considered in different points of the trajectory. Hence, when distance $d(t)$ becomes large, a new $d_0(t)$ is defined in order to evaluate the divergence, as follows:

$$\lambda = \frac{1}{t_n - t_0}\sum_{k=1}^{n}\log_b\left[\frac{d(t_k)}{d_0(t_{k-1})}\right] \tag{33}$$

NUMERICAL ANALYSIS OF THE NONLINEAR MECHANICAL SYSTEM

Initially, the aim is to verify that the mechanical system has nonlinear characteristics. In numerical applications, an Euler-Bernoulli beam model is proposed, supported by springs with linear and non-linear characteristics as shown in Fig. 1 and subjected to the application of a harmonic force. For the numerical discretization of the structure, the finite element method (FEM) is considered [39].

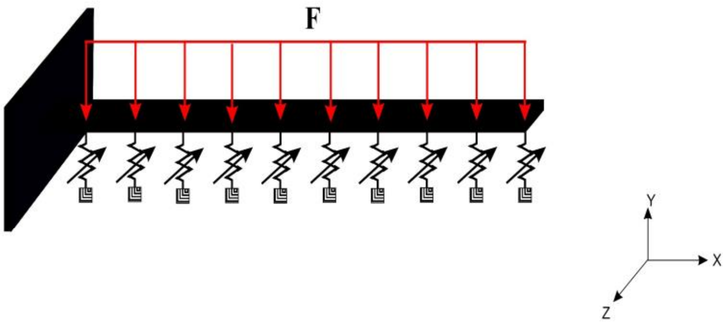

Figure 1. Scheme of the beam modeled in FE.

The x, y and z directions are considered to denote the dimensions of the beam geometry whose length, width and thickness are 0.5*m*, 0.01*m* and , respectively. The material physical properties consist of linear density ρ= $2710 \frac{kg}{m^3}$ and

$E = 7.10^{10}\,Mpa$, the longitudinal elastic module of the material from which the beam is made. The amount and position of the nonlinear springs are modified and the springs´ stiffness values are considered fixed:$k_{linear} = 100\, N/m^2$; and $k_{nonlinear} = 10^5\, N/m^2$. The numerical tests used the Newmark integration method. Using finite element procedures, the global matrices that make up the system´s equation of motion are obtained and the spring element is added to the stiffness matrix.
Thus, Eq. (1) is rewritten as:

$$[M]\{\ddot{x}(t)\}+[D]\{\dot{x}(t)\}+[K_0]\{x(t)\}=\{F(t)\} \tag{34}$$

where $[K_0]=[K]+[K_{linear}]$ and $[K_{linear}]$ is the stiffness matrix of the linear springs.

For most cases, the intensity variations of the exerted forces are very large, which shows a nonlinear relationship between stress and strain or between force and deformation. This fact instigates the use of springs with nonlinear characteristics. By coupling springs which have linear and nonlinear characteristics, Eq. (33) can be rewritten as follows:

$$[M]\{\ddot{x}(t)\}+[C]\{\dot{x}(t)\}+[K_0]\{x(t)\}+[K_{nonlinear}]\{x^3(t)\}=\{F(t)\} \tag{35}$$

where the term $[K_{nonlinear}]\{x^3(t)\}$ represents the contribution of the nonlinearity located in the system. In this case, a cubic nonlinearity is considered. The process for numerical analysis, implemented in the Matlab® environment, is described in Fig. 2.

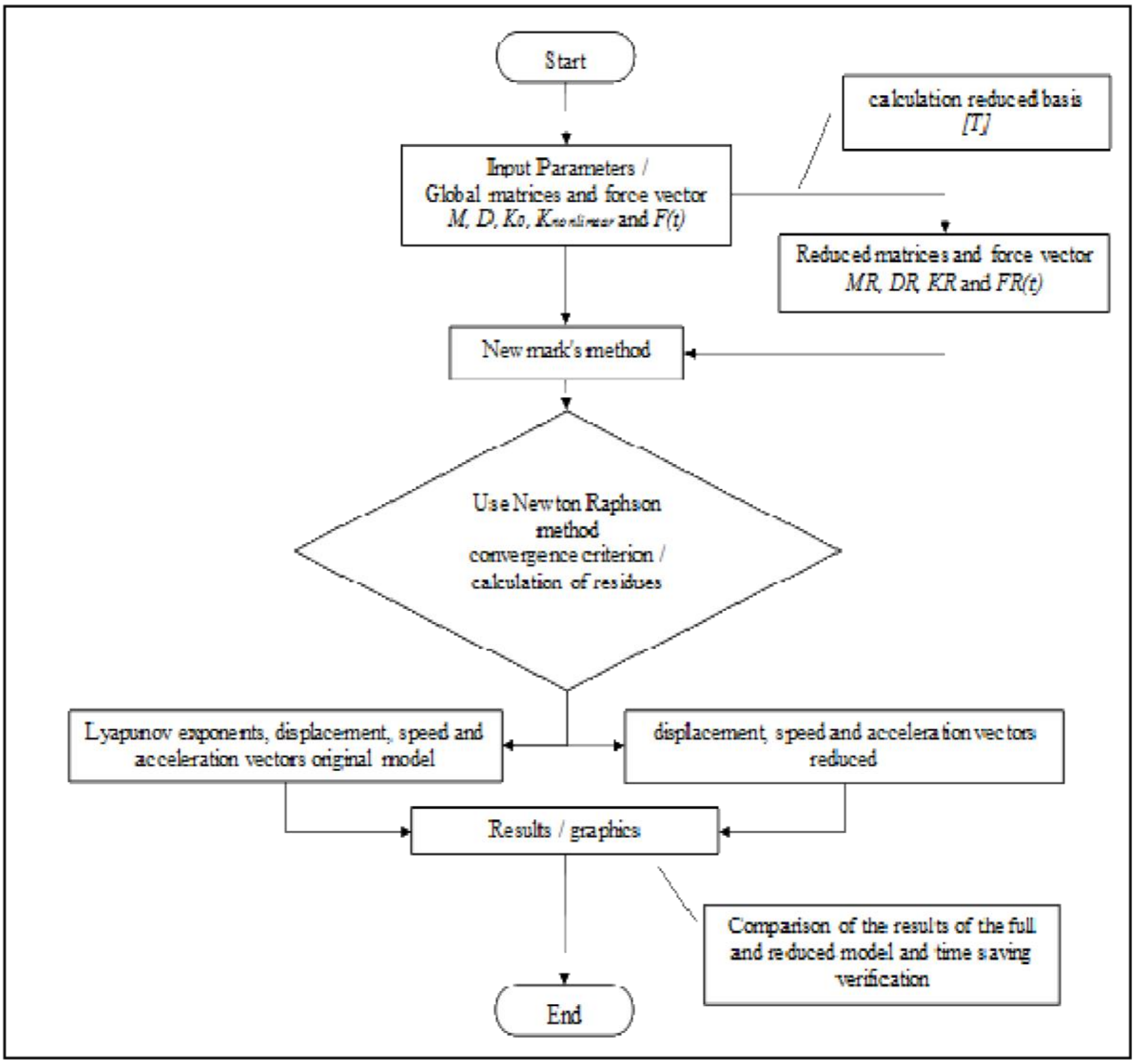

Figure 2. Flowchart of the computational implementation process.

Initially the global matrices of the system are calculated, the equation of motion is solved by numerical integration. The Newmark method together with the Newton Raphson method [11] are required to calculate displacements, velocities and acceleration. To verify the nonlinear behavior of the system, the phase space is represented and the Lyapunov exponents are determined.

Once the system has been characterized as chaotic or hyperchaotic, the process of obtaining the reduced models begins. The matrices of the system are required to obtain the transformation matrix in each method studied. The reduced equations of motion are solved, and the representativeness and time saved shown by using nonlinear reduced models can be observed.

Figure. 3 illustrates the dynamic response and phase space of the beam presented in Fig. 1. The beam is discretized in 10 elements and subjected to a nonlinear harmonic force of the form, $F_e(t) = F\sin(\omega t)$, where $F = 20N$ is the amplitude of the excitation and ω is the excitation frequency which coincides with the first natural frequency of the beam used. The numerical tests were performed in the time interval $t \in [t_0$

$= 0s\ t_f = 1\ ]s$, for an arbitrarily chosen constant time step, $\Delta t = 10^{-4} ms$, and the computations consisted of obtaining the normalized transverse displacements ($100 \times x(t)/length$).

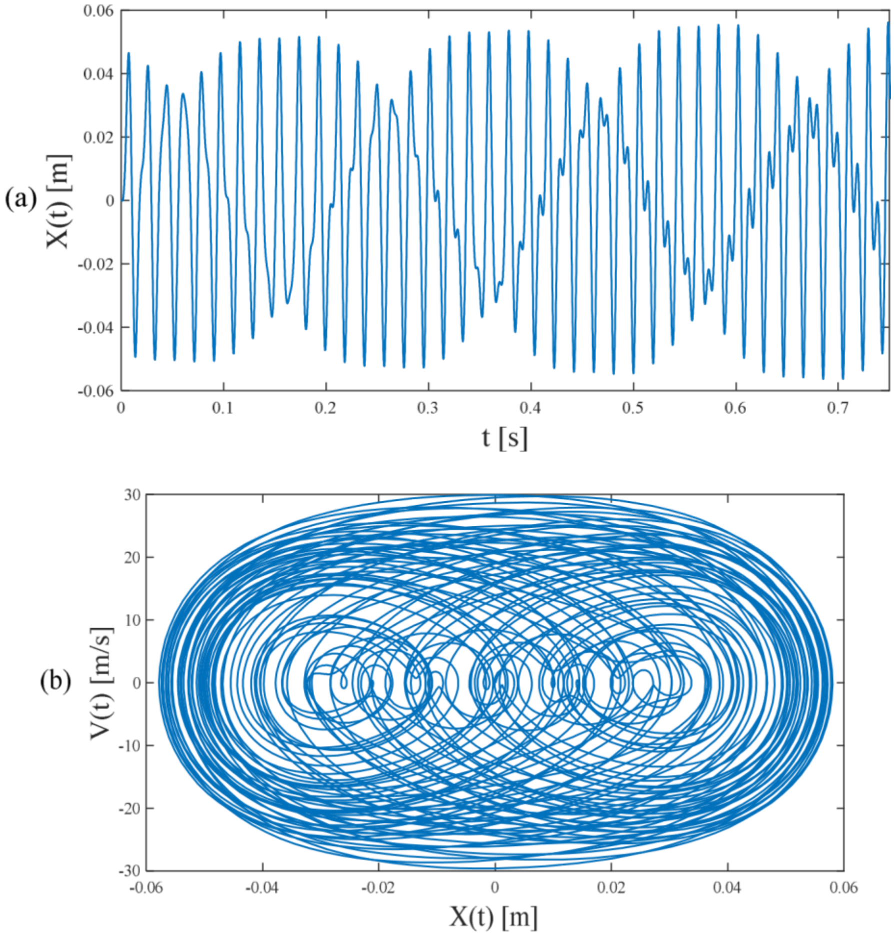

Figure 3. Dynamic responses (a) and phase space (b).

According to Fig. 3, the dynamic responses (a) and the phase space (b) show irregular oscillations due to the significant number of springs with non-linear characteristics, however, it is still not possible to say that there is a chaotic movement. Moreover, nothing can be said about the stability of the system and the presence of chaotic movements. The Lyapunov exponents are calculated and presented in Fig. 4.

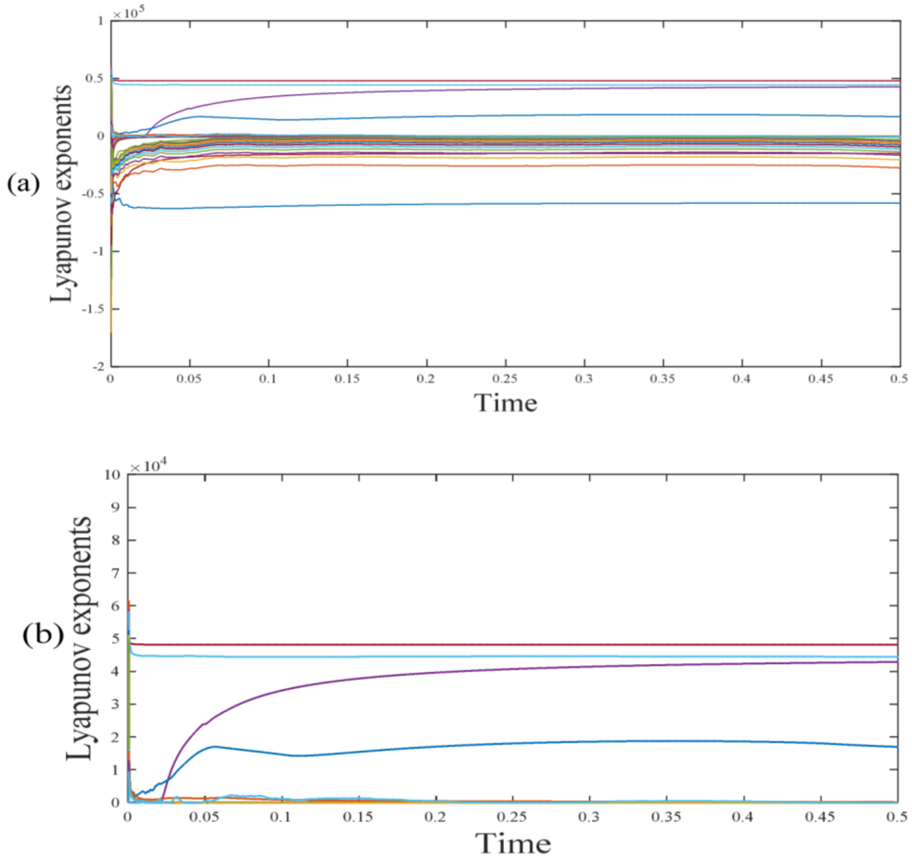

Figure 4. Evolution of the Lyapunov exponents (a) and zoom in the positive Lyapunov Exponent (b).

As shown in Fig. 4 (b), some of the Lyapunov exponent values were positive. Table. 1 presents these positive exponent values estimated in the suggested interval.

Table 1. Value of positive Lyapunov exponent.

Lyapunov Exponent (Λ)	Positive value
Λ_1	0.74
Λ_3	5.29
Λ_4	42981.95
Λ_7	48251.25
Λ_{13}	44540.95
Λ_{15}	17074.62
Λ_{16}	255.45
Λ_{17}	50.23

According to Fig. 4 and Tab. 1, for the interval considered, the system presents eight positive exponents, some show signs of conversion to zero if the interval was greater. However, in Fig. 4 (b) four of these exponents are well defined.

The presence of more than one positive exponent describes the system as hyperchaotic, i.e., it does not present conditions of regularity in the movement. This fact confirms the considerable influence of using springs with nonlinear characteristics in the responses of a dynamic system.

Once the nonlinearity in the system is identified, the efficiency of the model reduction methods in nonlinear systems can be verified. The base reduction matrices for each method are obtained and the complete model is reduced according to Eq. (3). For the system reduction process, we considered 12 degrees of freedom for the models reduced by the QUASI-GUYAN, SEREP and IIRS methods. For the EMB method, 12 modes of vibration and one enrichment vector were considered.

In Fig. 5, the responses estimated for the reduced models are compared with the responses of the complete model. According to the curves obtained in this figure, each method can be verified according to the commitment of quality and speed of response.

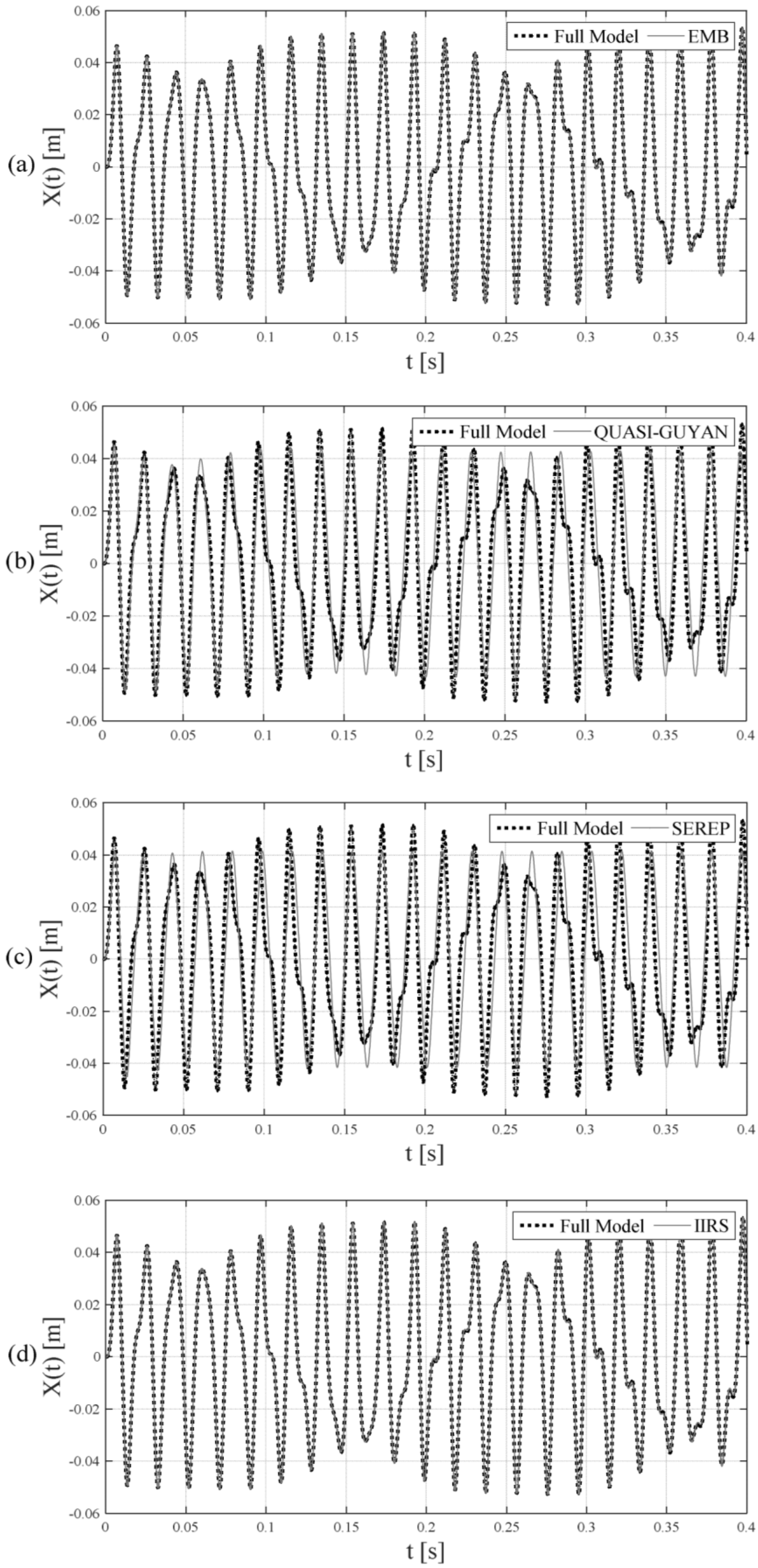

Figure 5. Dynamic response of the original model and reduced models (zoom), EMB (a), QUASI-GUYAN (b), SEREP (c), IIRS (d).

According to Fig. 5, the reduced models provided considerable quality of response. The EMB and IIRS methods presented higher results than those obtained by the QUASI-GUYAN and SEREP methods. This can be seen in Fig. 6.

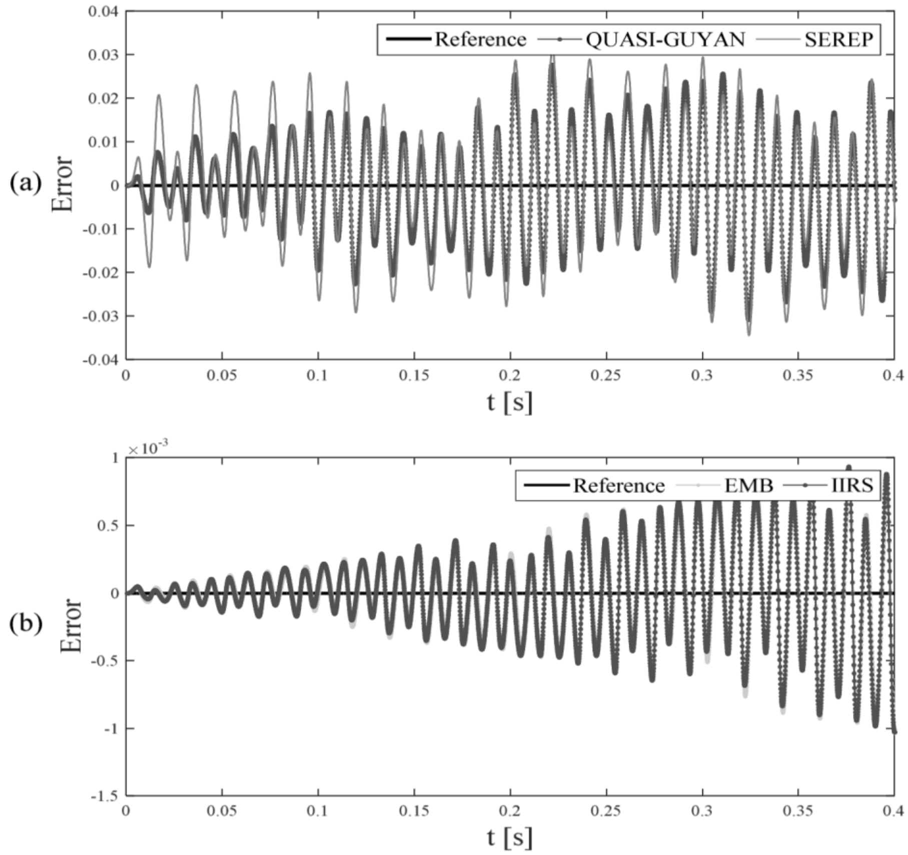

Figure 6. Percentage of relative error: QUASI-SEREP and SEREP (a); EMB and IIRS (b).

Figure. 6 shows the relative error percentage of the reduced models over time. According to the graphs, it can be observed that the QUASI-GUYAN and SEREP methods obtained equivalent answers with a high error percentage that implies that he low representativeness of the model. The relative error percentage of the EMB and IIRS methods show the effectiveness and robustness of these methods in terms of response quality, especially for the IIRS method that presented a smaller error.

It can also be observed that over the history of time that the percentage of error increases gradually over time. Thus, for a longer time interval, a new transformation matrix can be obtained.

In the validation of the reduced models as a tool to analyse non-linear systems, a reduction in the computational cost is considered, as well as the response quality, to obtain the results presented by each reduced model as a performance criterion. Figure 7 shows the percentage of time consumed by the reduced models in relation to the complete model.

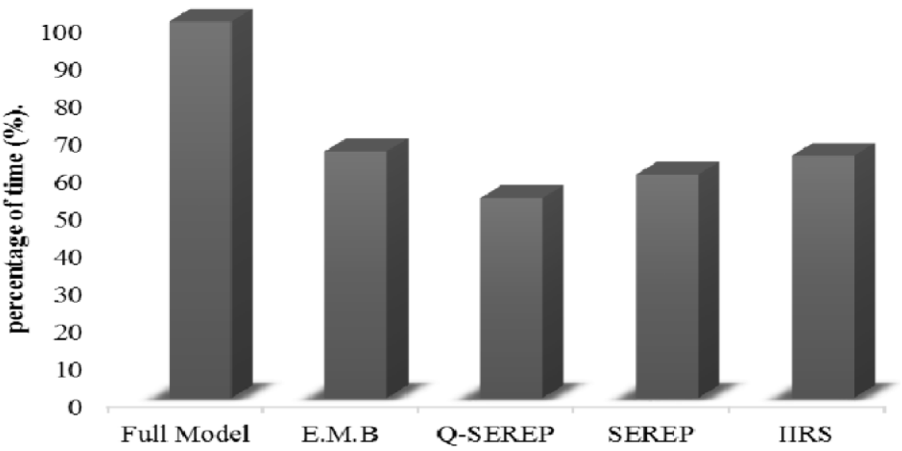

Figure 7. Computational time.

The speed of response presented by the reduced models is related to the complexity of the model. In the example of this work, due to the non-linearity, a plausible quantity of degrees of freedom was considered retained in the process of reduction in the search for a good representation. This fact directly influences the time saved shown in Fig. 7.

Still according to Fig. 7, the greatest time gains were provided by QUASISEREP and SEREP, however they did not obtain good representations. IIRS and EMB presented high computational costs, but they also obtained a higher response quality. Analyzing the two current criteria, the quality of response and time saved, the IIRS method proved to be better than the other approaches, with time saved of approximately 35%.

CONCLUSIONS

In this study, mechanical structures with nonlinearities were implemented using the finite element method. Procedures to calculate the Lyapunov exponents were proposed and these led to characterizing the studied system as hyperchaotic.

Using model reduction methods in the presence of nonlinearities can be observed by adopting two basic criteria: the quality and speed of response of the reduced models. Some methods were more robust, such as IIRS, which provided excellent response quality and time saved of approximately 35%.

The results show the relevance of adopting model reduction methods as a useful tool to analyze nonlinear systems discretized by the finite element method.

In the future, we intend to evaluate the performance of these methods in models of higher complexity, with different types of non-linearity, discretized by a significant number of finite elements.

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