Decision Theory under Uncertainty Mean-Variance Approach

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Abstract

In this paper, we mainly consider the theory and analysis of Decision under uncertainty which is making the foundations of all finance and portfolio theories. Decision makers face choices among a number of risky alternatives which is represented by lotteries. This paper develops alternative theories for choices under risk which expected utility theory which is derived from reasonable axioms about rational behavior in risky environment. An alternative theory of choice is developed, in which value is assigned to gains and losses rather than to final assets. All risky alternatives can be summarized by two numbers – mean u and variance σ^2 *called Mean-Variance Theory (MVT). This implies that typical mean-variance utility function v* $(\mu,\sigma) = \mu - \frac{\gamma}{2}\sigma^2$ *which is increasing in* μ *and decreasing in* σ*.The results show that, investors have different mean variance utility functions but the main results regarding optimal portfolio of risky assets do not depend on the specific utility functions of investors.*

Keywords: Uncertainty, decision, aversion, lotteries

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1. Introduction

Financial literature widely discusses the investment decisions of companies. This field has seen a surge of research with both theoretical and experimental advances. The study of the relationship between the decision under uncertainly and investment level is the most common way of analyzing the problems of over- and under risk. However, the study of Decision Theory under Uncertainty risk Mean-Variance Approach in financial distress is a topic that still requires more in-depth study. According to previous study, it identifies the existence of financial constraints as a key variable. This Implies that almost every decision we ever make in our lives, involves uncertainty. Uncertainty is then meant to represent "non probabilized" uncertainty –situations in which the decision makers is not given a probabilistic information about the external events that might affect the outcome of a decision–, as opposed to risk which is "probabilized" uncertainty. We assume that decision maker faces a choice among a number of risky alternatives which is represented by lotteries.

We will thus concentrate on situations in which to provide the application of decision theory to the problem of investment and making under uncertainty to the financial markets. The main contribution of this work is to find whether there is a conclusive correlation between the decision makings under uncertainty in the context of the financial, and to conduct a Mean-Variance Portfolio Analysis on over risky returns. In our research we present decisions under uncertainty, where Choice under uncertainty characterized as the maximization of expected utility. We develop the notion of MeanVariance Theory (MVT) as one representation of preference over risky.

The paper proceeds as follows. We start with a rather didactic section in which we present some definition lotteries to decision under uncertainly following some example.

In the three sections coming we develop alternative theories for choices under risk: Expected Utility Theory, Mean-Variance Theory, and Prospect Theory. We then discuss in Section 6 Mean Variance Portfolio Analysis and Capital Asset Pricing Model (CAPM) to end this section.

2. Theory

This survey is concerned with introduction to some of the most important definitions and examples used within the decision Theory under Uncertainty.

Almost every decision we ever make in our lives, involves uncertainty, i.e. the outcome of our choices cannot be predicted with absolute certainty. These decisions include not only financial investment choices, but career choices, marriage, and college major. Decision theory under uncertainty makes the foundations of all finance and portfolio theories. We assume that decision maker faces a choice among a number of risky alternatives. Each risky alternative may result in one of a number of possible outcomes, but which outcome will actually occur is uncertain at the time of decision making. We represent these risky alternatives by lotteries.

Definitions and examples

These definitions that we are going to tell, we help us understand the notion of lotteries and how it uses by following examples to explain the development alternative theories for choices under risk. By contrast, uncertainty represents a situation in which no probabilistic information is available to the decision maker.

What is a lottery?

A lottery is a probability distribution defined on a set of payoffs. It can be discrete, in which case it is described by the list of payoffs $x_1; x_2; :::; x_N$ and the probabilities of these payoffs $p_1; p_2; :::; p_N$. The number of outcomes in a discrete lottery can be finite $N < 1$, or infinite. A lottery can also be continuous, in which case the set of payoffs is usually a subset of real numbers ($x \subset$ R) and the distribution is described with a probability density function *(pdf)* $f(x)$ or the cumulative distribution function *(cdf)* F(x) $=\int_{-\infty}^{x} f(t)dt$.

Lotteries can also be a mix of discrete and continuous, but we will not deal with such lotteries in this thesis. We use the notation $\mathcal{L}$ to denote the space of all possible lotteries.

Just like in the case of any probability distribution of a random variable, we require that

[Discrete] : $p_i \geq 0 \quad \forall\, i, \sum_i p_i = 1$

[Continuous] : $f(x) \geq 0 \quad \forall\, x, \int_{-\infty}^{\infty} f(x)dx = 1$

Just like with any random variables, we might want to compute the mean and variance of a lottery, and also the covariance between any two lotteries.

In general, lotteries can have any kind of abstract outcomes, not only monetary payoffs. For example, when you play basketball, you can win, loose, end with overtime or have an injury. These outcomes are not stated in monetary terms. For simplicity of analysis, we assume that these outcomes can be represented in terms of money, so a win is for example equivalent to +\$500, and a loss is equivalent to -\$400.

Moreover, money lotteries are all we need for the study of financial decisions, where outcomes are naturally represented with monetary returns. Also note that sure outcomes can also be viewed as lotteries that have one outcome occurring with probability 1.

Example 1: A discrete lottery A, with two outcomes, \$1000 and \$500, each achieved with equal probabilities.

$$A = \begin{cases} \$1000 \text{ w.p. } 1/2 \\ \$500 \text{ w.p. } 1/2 \end{cases}$$

The expected value and the variance of this lottery are:

$E(A) = 1000 \times 0.5 + 500 \times 0.5 = \$750$

$Var(A) = (1000 - 750)^2 \times 0.5 + (500 - 750)^2 \times 0.5 = 62.500$

This example is not supposed to be realistic but to show how to determine The expected value and the variance of discrete lottery.

Let consider A; B ∈ $\mathcal{L}$ be lotteries and ∈ [0; 1]. Then $C = \lambda A + (1-\lambda) B$ denotes a compound lottery where with probability the lottery A is played and with probability (1-) the lottery B is played. If the lotteries are discrete, then the probability of outcome i in C is $P_i^C = P_i^A + (1 - \ )P_i^B$, and when the lotteries are continuous, the probability density of C is $f_C(x) = \ f_A(x) + (1- \ ) f_B(x)$.

To be clear of our example the main challenge of decision theory is to define preference over lotteries that will allow comparison of different lotteries. For example, consider a

lottery D, which pays \$750 with probability 1. Which lottery is better, A or D? This question is equivalent to asking "would you be willing to pay \$750 for lottery A? More generally, how much are you willing to pay for lottery A? This is the fundamental question of asset pricing. Most people will say that they would pay less than \$750 for lottery A.

In general, most people are not willing to pay the mean of a lottery to buy that lottery.

This suggests that things other than the mean of a lottery are also important. Nevertheless, until recently (mid-20th century), the only theory of preferences over risky alternatives was the mean theory - i.e. the value of a lottery is given by its mean payoff. The next example illustrates the first challenge to the mean theory. It is known as the "St. Petersburg Paradox", analyzed by the Swiss mathematician Daniel Bernoulli in 1738. Let take another example.

Example 2**:** We consider the lottery A, based on the following gamble.

A fair coin is tossed repeatedly, until "tails" first appears. This ends the game. Let the number of times the coin is tossed until "tails" appears be k. The lottery pays $\$2^{k-1}$. Thus, if you toss the coin once, and "tails" appear, then you are paid $2^{1-1} = 1$. If it takes 5 tosses until the coin shows "tails", then you are paid $2^{5-1} = 2^4$

= \$16. Thus, the possible payoffs of this lottery are 2^{k-1}, k = 1; 2; …, and the probabilities are $(1/2)^k$, k = 1; 2;… The expected value of this lottery is: $E(A) = \sum_{k=1}^{\infty} 2^{k-1} \left(\frac{1}{2}\right)^k = \frac{1}{2}\sum_{k=0}^{\infty} 2^k \left(\frac{1}{2}\right)^k = \frac{1}{2}\sum_{k=0}^{\infty} 1 = \frac{1}{2}\,\infty = \infty.$ Once again, the question is, how much are you willing to pay for this lottery? Despite the fact that the expected payoff of this lottery is , there is not a single person who would pay \$ to play this lottery.

In fact, most people are willing to pay no more than a few dollars to play this lottery. Indeed, this game can give you a high payoff of more than a million dollars, if you toss the coin 21 times before "tails" first appears (your payoff in this case is 2^{20} = \$1 048 576). But this happens with very low probability $\left(\frac{1}{2}\right)^{21}$ = 0.000000476837, or once in more than 2 million plays. Example 2 (St. Petersburg Paradox) demonstrates once again that, in general, the value of a lottery is not equal to the expected value of its payoff. Nevertheless, until middle of the 20th century, the expected value was the well-accepted theory of decisions under risk.

In the next three sections we develop alternative theories for choices under risk: (i) Expected Utility Theory, (ii) Mean-Variance Theory (MVT), and (iii) Prospect Theory. We start with the main-stream theory in economics - the Expected Utility Theory (EUT).

3. Expected Utility Theory (EUT)

Expected utility theory has dominated the analysis of decision making under risk.

According to the last examples, the motivation for the development of EUT is Example 2 (St. Petersburg Paradox). Bernoulli realized that twice the money is not always "twice as good". If a person has only a small amount of money, say \$1000, then doubling it increases his utility by more than say doubling the wealth of someone who has 10 million dollars. Put in another way, the marginal utility from money is diminishing - the more money you have, the smaller is the gain from additional \$1. This idea of diminishing marginal utility from money is equivalent to risk aversion in EUT, and will be formalized later. For now, Bernoulli’s intuition is that instead of computing the expected payoff of a lottery, we need to compute the expected utility of a lottery. The strength of EUT is that it is not only intuitively appealing, but can be derived from more fundamental axioms about preferences.

3.1. Axiomatic foundations of Expected Utility Theory

Savage (1954) book is still considered today to be one of the major achievements in decision theory. With a scarcity of input, he delivers the classic subjective expected utility representation of preferences. Savage thus ties together the idea of subjective probability advocated by Ramsey and de Finetti with the idea of expected utility derived (with given probabilities) by von Neumann and Morgenstern.

The starting point for any decision theory under risk is lotteries, and the assumption that people have some preferences over the space of all lotteries. We assume that there exists a weak preference relation $\succsim$ on L, such that for two lotteries A, $B \in \mathcal{L}$, the notation A$\succsim$B means that "lottery A is at least as good as lottery

B". From the weak preference relation, we can derive the strict preference relation and the indifference relation as follows. The strict preference relation $\succ$ on L means that A $\succ$ B if A $\succsim$ B but not B$\succsim$A. We read A~ B as "lottery A is strictly better than lottery B". The indifference relation~ on L means A ~ B if A $\succsim$ B and B $\succsim$A. We read A ~ B as "lottery A is as good as (or equivalent to) lottery B".

Next, we make some assumptions (axioms) about the preference relation$\succsim$, that will allow representing it with a utility functional and enable practical usage. We describe Savage's core axioms. The most important one is often referred to as the "sure thing principle".

A preference relation $\succsim$on $\mathcal{L}$ is called rational if it satisfies the following two axioms:

A1. Completeness: A preference relation $\succsim$ on $\mathcal{L}$ is complete if for any two lotteries *A*, *B* ϵ $\mathcal{L}$, either $A \succsim B$ or $B \succsim A$ or both. Completeness means that the decision maker is able to choose among risky alternatives.

A2. Transitivity: A preference relation $\succsim$on $\mathcal{L}$ is transitive if for any three lotteries *A, B, C* $\epsilon\mathcal{L}$, we have $A\succsim B$ and $B\succsim C$ $\rightarrow$ $A\succsim C$

The transitivity assumption is a natural consistency of preferences. We can show that if transitivity is violated for some individual, i.e. his preferences are A $\succsim$B and B$\succsim$C and C $\succ$ A, then we can easily extract all his wealth by offering him to trade B for C, A for B, C for A, and repeat many times, and each time he gets C for A he pays some amount (because C is strictly better than A).

The next assumption is technical, and is needed in order to ensure representation of preferences with utility.

A3. Continuity: The weak preference relation$\succsim$on $\mathcal{L}$ is continuous if for any lotteries A, B, C ϵ $\mathcal{L}$ with A $\succsim$ B $\succsim$C, there exists a probability p ϵ [0; 1] such that $B \sim pA + (1 - p)\, C$

The right hand side is the compound lottery, where with probability *p* lottery A is played and with probability *1 - p* lottery C is played. Here the term continuity means that any lottery "in between" two other lotteries (here B is in between A and C) is equivalent to some mixture of the two lotteries. What it also means is that there are no "jumps" in preferences due to small changes in probabilities.

For example, suppose that A is "basketball game with friends", and B is "staying at home", and C is a "knee injury", and I prefer playing basketball over staying at home: A$\succ$ B. Then, when I add a small enough probability of a knee injury to the basketball game, I would not suddenly change my mind and decide to stay at home. That is, B does not become better than $pA + (1- p)C$ for small enough $(1- p)$. This axiom is reasonable because most people do go out to work, shopping, movies, despite the small risk of getting into accident.

Let $\succsim$ on $\mathcal{L}$ be a weak preference relation on the space of lotteries. If $\succsim$ satisfies the axioms A1, A2 and A3, then there exists a continuous utility functional $U\colon \mathcal{L} \rightarrow R$ such that $U(A) \geq U(B)$ if and only if $A\succsim B$ for any lotteries $A, B\ \epsilon\mathcal{L}$. This also implies that, for any lotteries, U (A) > U (B) if and only if $A \succ B$ and $U(A) = U(B)$ if and only if $A\sim B$.

We say that a utility functional $U\colon \mathcal{L} \rightarrow R$ has expected utility form if there exists a utility function $u : R \rightarrow R$, such that for every lottery L$\epsilon\mathcal{L}$, $U(L) = E[u(x)]$

In words, the utility of a lottery is equal to the expected utility of its payoffs. Specifically, for discrete lottery

$$U(L) = E[u(x)] = \sum_i u(xi)\, pi$$

and for continuous lottery (with pdf *f (x)*)

$U(L) = E[u(x)] = \int_{-\infty}^{\infty} u(x)\, f(x)\, dx$

The utility function *u* is sometimes called "von Neumann-Morgenstern" (vNM) utility function, after the mathematician John von Neumann (1903 .1957) and the economist Oskar Morgenstern (1902 - 1977) who provided the axiomatic foundations for the EUT. These two are also the founders of Game Theory. We can conclude by this tendency the advantage of EUT is that it is derived from reasonable axioms about rational behavior in risky environment. Observe that a key advantage of the EUT is that in order to know the preferences over risky alternatives, all we need to know is the preferences over sure outcomes, given by the vNM utility function $u: R \rightarrow R$. In all applications, we make the standard assumption that *u* is increasing, which means that more money is better (formally monotonicity assumption).

A4. Independence of irrelevant alternatives. A, B $\epsilon \mathcal{L}$ two lotteries with $A \succ B$, and let ϵ (0;1]. Then for any lottery $C \epsilon \mathcal{L}$, it must be $\lambda A + (1-\lambda) C \succ \lambda B + (1-\lambda) C$

The independence axiom means that the ranking of two lotteries does not change if you mix each of them with a third lottery. Preferences that satisfy the independence axiom, in addition to completeness, transitivity and continuity, can be represented with expected utility form.

The certainty equivalent (CE) of a lottery L is the non-random payoff CE which is equivalent to playing the lottery: CE ~ L. If preferences can be represented by expected utility, then certainty equivalent is defined by *u* (CE) = *E [u (x)]*.

Suppose that the preference relation $\succsim$ on $\mathcal{L}$ has expected utility representation with vNM utility function *u*, then, $v(x) = a + bu(x)$ is another vNM utility function representing the same preferences as *u*.

We need to show that for any two lotteries $A, B \epsilon \mathcal{L}$. To do that we need to know first

$E_A[u(x)] \geq E_B[u(x)] \leftrightarrow E_A[v(x)] \geq E_B[v(x)]$

Where the subscripts *A* and *B* indicate that we are computing the expected values using the probability distributions *A* and *B*. Note that we can write:

$E_A[v(x)] = E_A[a + bu(x)] = a + bE_A[u(x)]$ *first*

$E_B[v(x)] = E_B[a + bu(x)] = a + bE_B[u(x)]$ *second*

Subtracting the second from the first:

$E_A[v(x)] - E_B[v(x)] = b\{E_A[u(x)] - E_B[u(x)]\}$

All expectations are numbers, and since b > 0 we have:

$E_A[v(x)] - E_B[v(x)] \geq 0 \leftrightarrow E_A[u(x)] - E_B[u(x)] \geq 0$

3.2. Risk Aversion

We have seen in example 1 a lottery that pays $1000 or $500 with equal probabilities,

$$A = \begin{cases} \$1000 \text{ w.p. } 1/2 \\ \$500 \text{ w.p. } 1/2 \end{cases}$$

The expected value of this lottery was found to be *E (A) = $750*, and we mentioned that most people would not pay $750 for this lottery. In general, most people prefer the mean of a lottery for sure, over the lottery itself. This behavior is called risk aversion.

We call a person risk-averse if he prefers the expected value of every lottery $L \in \mathcal{L}$ over the lottery itself, i.e. $E(L) \succ L$. If preferences over lotteries can be represented with expected utility form, then risk aversion means $u[E(x)] > E[u(x)]$.

A person is risk-seeking if he prefers every lottery over its expected value, $L \succ E(L)$. A person is risk-neutral if he is indifferent between lotteries and their expected return, $L \sim E(L)$. Once again, if preferences over lotteries can be represented with expected utility form, then risk-seeking means $E[u(x)] > u[E(x)]$, and risk-neutral means $u[E(x)] = E[u(x)]$. Let ;ore detail, a person described by EUT is:

(i) risk-averse if and only if his vNM utility function u is strictly concave, (ii) risk-seeking if and only if his vNM utility function u is strictly convex,

(iii) risk-neutral if and only if his vNM utility function u is linear.

This is simple principle to understand. We need to show that $u[E(x)] > E[u(x)]$ for any lottery, any tangent line to the graph of a strictly concave function, lies above the graph of the function. Figure1.1 illustrates this graphically. In particular, the figure shows a tangent line $g(x)$ at the point $(\mu; u(\mu))$, where $\mu \equiv E(x)$ is the mean of a lottery.

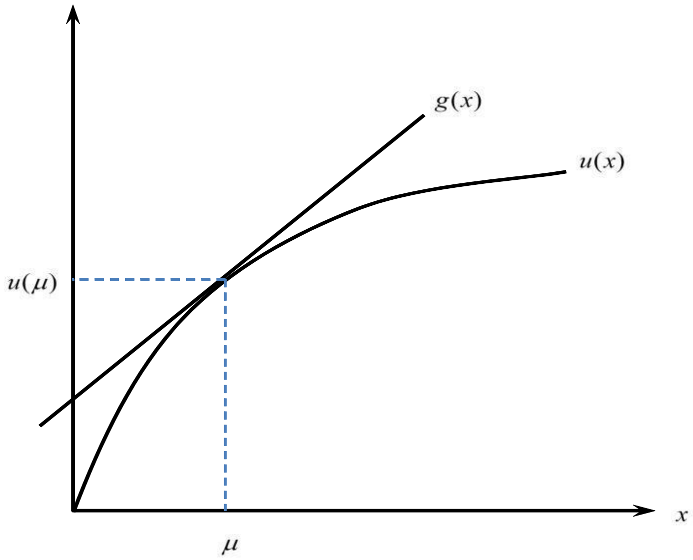


Figure1: Strictly concave u and a tangent line at $(\mu, u(\mu))$

The equation of the tangent line is $g(x) = u(\mu) + u'(\mu)(x - \mu)$, and since it lies above $u(x)$ for all x with the exception of x = μ, we have $\forall x \neq \mu$ $g(x) > u(x)$

$u(\mu) + u'(\mu)(x - \mu) > u(x)$

Taking expectation $u(\mu) + u'(\mu)[E(x) - \mu)] > E[u(x)$ $u(\mu) + u'(\mu)[\mu - \mu)] > E[u(x)]$ and finally we obtain $u[E(x)] > E[u(x)]$.

3.3. Measuring risk aversion

We have established that risk-averse individuals, whose preferences can be represented with expected utility functional, must have concave vNM utility function u. Some question is asking here according to this section which utility functions are suitable? Different individuals might have different degree of risk aversion, so how do we capture these differences with the utility function? The next definition introduces two ways of measuring the degree of risk aversion.

Given a twice differentiable vNM utility function $u: R \rightarrow R.$, The Arrow-Pratt coefficient of absolute risk aversion at *x* is defined

$$ARA: = -\frac{u''(x)}{u'(x)}$$

The Arrow-Pratt coefficient of relative risk aversion at *x* is defined

$$RRA: -\frac{u''(x)}{u'(x)}x =$$

Intuitively, the "more concave" the utility function is, the greater should be the degree of risk aversion. Therefore, both measures have the second derivative, which is supposed to capture the degree of concavity. The minus in front makes both measures positive numbers (since the second derivative is negative for strictly concave functions).

The both have $u'(x)$ in the denominator to eliminate the effect of multiplication by a positive constant.

The different between two measures is the absolute risk aversion is relevant for choices involving absolute gains and losses from current wealth, while the relative risk aversion is relevant for choices involving percentage (or fractional) gains or losses of current wealth.

What we need to know here is the Arrow-Pratt measures of degree of risk aversion allow us to establish whether one individual is more risk-averse than another individual. That is, individual 2 is more risk averse than individual 1 if for every x we have $ARA_2(x) > ARA_1(x)$ or $RRA_2(x) > RRA_1(x)$ for $x > 0$. There are several other, equivalent ways, of making the same comparison across individuals. As we know about risk-averse, another risk called risk premium *RP* is important to know in this section.

A risk premium *RP* for a given lottery is the amount that an individual is willing to pay out of the expected payoff in order to avoid playing the lottery, and instead receives its mean with certainty: $u[E(x) - RP] = E[u(x)]$

Notice that on the left side we have utility from certainty equivalent. That is, the risk premium is just the difference between the mean of a lottery and its certainty equivalent, i.e. $RP = E(x) - CE$. Therefore, an individual is more risk-averse if the risk premium he is willing to pay for avoiding any lottery is higher.

Yet another equivalent way to compare individuals according to their degree of riskaversion is by letting the more risk-averse vNM utility function be "more concave". In other words, the vNM utility function u_2 is more risk averse than u_1 if u_2 is obtained through an increasing and concave transformation of u_1. $u_2(x) = \upsilon(u_1(x))$ for some increasing and concave function $\upsilon(.)$.

The different definitions of risk aversion and of more risk aversion help us prove some testable predictions about the behavior of risk-averse individuals. For example, even risk averse individuals will invest some of their wealth in risky assets, provided that the return of the risky assets is high enough. However, the amount invested in risky assets is smaller for more risk-averse individuals.

4. Mean-Variance Theory

According to the previous section expected utility theory has several advantages. First, it is derived from precise axioms about human behavior, so users of the theory know exactly what assumptions about preferences make the EUT valid. Second, the expected utility theory helped us understand why people are usually unwilling to pay for a lottery the expected value of its payoffs - a behavior known as risk aversion. EUT explains why people buy insurance, while at the same time invest in risky assets. We have seen applications of EUT to optimal investment and demand for insurance. Notice however that in choosing optimal investment in risky asset, we assumed that the entire distribution of returns is known.

In a more realistic setting, of choosing optimal portfolio with many risky assets, we would need to know (or estimate) the joint distribution of all asset returns - a monumental task.

This is why in 1952 Harry Markowitz introduced a theory of portfolio selection based on the simplifying assumption that all risky alternatives can be summarized by two numbers – mean *u* and variance σ^2 - the Mean-Variance Theory (MVT).

In the next section we will learn the details of Markowitz portfolio selection theory, but in this section we discuss the assumptions behind the mean-variance analysis. In particular, how good is the key assumption of the MVT, that in evaluating risky alternatives people only care about the mean and variance of the returns? We also ask, under what assumptions the MVT is as valid as the EUT.

We say that a utility functional $u: \mathcal{L} \rightarrow R$ has mean-variance form if there exists a utility function $u: R \times R_+ \rightarrow R$, such that for every lottery L $\epsilon \mathcal{L} U(L) = u(\mu, \sigma^2)$ Where

$\mu = E(L)$ and $\sigma^2 = Var(L)$, and such *u* is called the mean-variance utility function. Equivalently, the mean-variance utility function can be written as $u(\mu, \sigma)$, i.e. as a function of mean and standard deviation, instead of variance.

In other words, the utility derived from any lottery depends only on the mean and variance of that lottery. Notice the notation $u: R \times R_+ \rightarrow R$ which means that the meanvariance utility function *u maps* elements from the two dimensional Euclidian space into real numbers.

The second dimension is restricted to non-negative real numbers because variance cannot be negative. Thus, the mean-variance utility function maps vectors (μ, σ^2) into numbers. The mean-variance utility function is therefore very different from the vNM utility function that maps single numbers (quantities of money or payoffs) into real numbers. But there are some similarities between the mean-variance utility functions and the vNM utility functions. Similar to monotonicity (more is better) of the vNM utility function *u*, which requires that it is an increasing function, the mean-variance utility is assumed to be increasing in the meanμ. A lottery with higher mean is better, ceteris paribus (for the same variance). We will always assume that *u* is strictly monotone.

Using mean-variance utility, one can define risk aversion in the standard way, just like in EUT, where risk-averse individual is one who prefers the mean of any lottery over the lottery itself. A mean-variance utility function $u: R \times R_+ \rightarrow R$ is called risk-averse if $u(\mu, 0) \geq u(\mu, \sigma)$ for all σ and all $\mu > o$ Similarly, an individual is risk-seeking if $\sigma > 0 \rightarrow u(\mu, 0) < u(\mu, \sigma)$ while risk-neutrality means that $u(\mu, 0) = u(\mu, \sigma)\ \forall \sigma$.

The next definition is a bit stronger assumption than risk aversion.

A mean-variance utility function $u: R \times R_+ \rightarrow R$ is called variance-averse if $u(\mu, \sigma_1) \geq u(\mu, \sigma_2)$ for all μ and all $\sigma_1 < \sigma_2$. It is strictly variance-averse if $\sigma_1 < \sigma_2 \rightarrow u(\mu, \sigma_1) \geq u(\mu, \sigma_2)$.

The applications of the mean-variance theory are vast, especially to the portfolio analysis.

We now turn to the critique of the mean-variance theory.

4.1. Critique of the mean-variance theory

One problem that is very obvious is that the mean-variance theory cannot always be applied to all lotteries. Mean-variance representation is not grounded in such axioms, and it is not immediately clear what kind of preferences can be represented with mean variance utility.

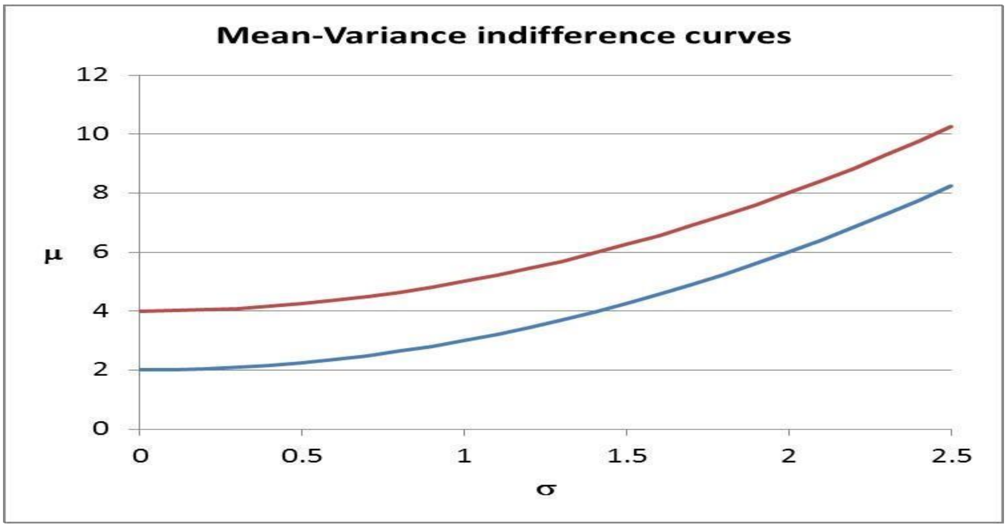


Figure 2: Mean-variance indifference curves.

4.2. Validity of the mean-variance theory

We discuss three cases when the mean-variance theory arises as a special case of the expected utility theory:

1. Quadratic vNM utility function:

Let $\succsim$ be a preference relation on $\mathcal{L}$ which can be described by expected utility with vNM utility function *u*. If *u* is quadratic, then there exists a mean-variance utility function *v* $(\mu\ \sigma,)$ which also describes $\succsim$.

2. Normally distributed payoffs:

Let $\succsim$ be a preference relation on $\mathcal{L}$ which can be described by expected utility with vNM utility function *u*. Suppose that payoffs are normally distributed, i.e. $x \sim N(\mu, \sigma^2)$, with pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right], -\infty < x < \infty$$

Then, there exists a mean-variance utility function $v(\mu, \sigma^2)$ which describes $\succsim$ for all normal lotteries.

3. Small risks

Suppose that you have initial wealth *w*, to which we add a small risk *z*, with $E(z)=0$ and $Var(z) = \sigma^2$, so the total wealth is random: $w + z$. We can always make the assumption that $\mu_z = 0$, because if the risk was *y* with $\mu_y \neq 0$, we could write the resulting wealth as $w + \mu_y + y - \mu_y$, and then redefine the initial wealth as $w + \mu_y$ and the mean-zero risk $z = y - \mu_y$. Therefore, the mean and variance of the final wealth are:

$E(w + z) = w + E(z) = w = \mu$

$Var(w + z) = Var(z) = \sigma^2$

To summarize, in this section we introduced the widely used in practice Mean-Variance Theory. Its main advantage is simplicity, and the main drawback is that it violates the First Order Stochastic Dominance. However, we have shown that under certain special cases, the MVT is equivalent to the EUT: (i) when vNM is quadratic, (ii) when payoffs have normal distribution or any distribution completely determined by mean and variance, (iii) when risks are small.

5. Prospect Theory

Recall that Expected Utility Theory is grounded in axioms of what is considered rational behavior. In that sense, the EUT is prescriptive - it prescribes how people should (or supposed to) behave when choosing among risky alternatives. But does the EUT do a good job describing how people actually behave? In many situations the EUT does describe actual behavior pretty well. Nevertheless, there are plenty of experimental and real world examples where the EUT fails. The development of Prospect Theory (Daniel Kahneman and Amos Tversky 1979), and the Cumulative Prospect Theory (Daniel Kahneman and Amos Tversky 1992) is an attempt to improve on the EUT in order to explain actual behavior with risky alternatives. Thus, Prospect Theory is descriptive attempts to describe actual behavior.

5.1. Motivation

The key ingredients of the different version of Prospect Theory are: framing, and probability weighting. Framing refers to the way the particular choices are formulated (framed), and the reference for gains and losses. Probability weighting is the observed tendency of people to consistently inflate low probabilities, and deflate high probabilities.

Example 3 (Framing).

Imagine that your country is preparing for the outbreak of a deadly virus, which is expected to kill 600 if no cure is found. You need to choose between two programs: A, B. With program A, 200 people will be saved. With program B there is a 1/3 probability of saving all 600 people, and 2/3 probability of not saving anyone.

Thus, when the choice is framed in terms of numbers of people saved, the two alternatives are:
$$A = \left\{ +200 \quad w.p.\ 1 \right., \qquad B = \begin{cases} +600 & \text{w.p. } 1/3 \\ 0 & \text{w.p. } 2/3 \end{cases}$$

The majority (72%) of doctors who participated in this experiment chose A over B. This is indeed what the EUT predicts that risk-averse individuals should choose. The mean of B is 200, so risk-averse individuals should prefer the mean of this lottery (i.e. program A) over the lottery itself. In a different experiment, with another group of doctors, the participants had to choose between programs C and D. If program C is adopted, 400 people will die with certainty, and if program D is adopted, there is 1/3 probability that nobody will die, and 2/3 probability that 600 will die.

Thus, framed in terms of people dead, the two alternatives are:

$$C = \left\{ -400 \quad w.p.\ 1 , \qquad D = \left\{ \begin{array}{l} 0 \ \text{ w.p. } 1/3 \\ -600 \text{ w.p. } 2/3 \end{array} \right. \right.$$

The majority of doctors who participated in this experiment (78%) chose D over C. Here the EUT predicts that risk-averse individuals should again choose the mean of a lottery over the lottery. The mean of D is -400, so the EUT predicts that risk-averse individuals will choose C over D, which is the opposite from what the majority of participants chose. Thus, it seemed like the participants in this experiment were risk-seeking.

Notice that A is the same as C, while B is exactly the same program as D. The difference is in the way these programs are framed. A and B are framed in terms of lives saved (gains) and C and D are framed in terms of numbers of dead (losses). Thus, Kahneman and Tversky modified the original vNM utility function so that it is consistent with riskaverse behavior for gains (concave), and risk-seeking behavior for losses (convex).

Figure 3 describes such utility function, which we call value function, and denote by $v:R \rightarrow R$ As in EUT, we will always assume that the value function v is monotone increasing.

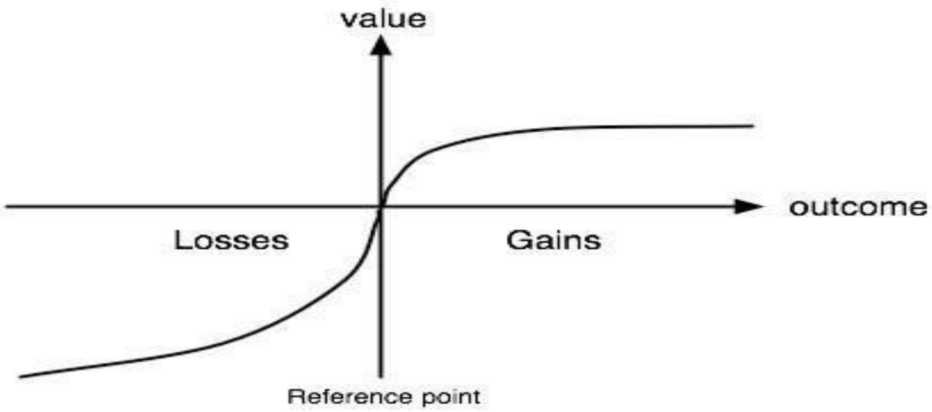


Figure 3: Prospect theory value function.

Notice that the vNM utility function $u:R \rightarrow R$ in EUT, was not restricted to be concave or convex, and it potentially could have the shape of the Prospect Theory value function in figure 1.3. However, the EUT is usually applied to final wealth, which is usually positive.

The prospect theory however abstracts from wealth, and focuses on gains and losses. But nevertheless, the above value function seems more like a refinement of the EUT, rather than an altogether new theory.

To summarize there are many versions of Prospect Theory (PT) today, but they all share the properties of (i) framing, and (ii) probability weighting.

 Original Prospect Theory 1979.

For any lottery $A \in \mathcal{L}$, with outcomes $x1 < x2 < ... < xn$ and probabilities $P_1; P_2;.. ; Pn$, and given the individual's value function $v: R \rightarrow R$ and his weighting function $w: [0; 1] \rightarrow [0; 1]$, the Prospect Theory utility from the lottery A is given by

$_{PT}(A) = \sum_{i=1}^{n} w\ (p_i) v\ (x_i)$

This looks similar to EUT, except that the vNM utility function *u* is replaced by the prospect theory value function *v*, and instead of the actual probabilities of outcomes p_i, the individual uses the weights $w\ (p_i)$.

Therefore, the original prospect theory was modified, and in 1992 Kahneman and Tversky developed the so called Cumulative Prospect Theory (CPT).

 Cumulative Prospect Theory 1992.

For any lottery $A \epsilon \mathcal{L}$, with outcomes *x1 < x2 <... < xn* and probabilities $P_1; P_2;.. ; Pn$, and given the individual's value function *v: R →R* and his weighting function *w: [0; 1] → [0; 1],* the Cumulative Prospect Theory utility from the lottery A is given by

$_{CPT}(A) = \sum_{i=1}^{n}\ [w\ (F_i) - w\ (F_{i-1})] v\ (x_i)$

This seems like a very similar definition to the original Prospect Theory, and indeed in many cases the resulting utility from a lottery is very similar (PT (A) is similar to CPT (A)). The prospect theory (PT or CPT) is more general than the EUT, and seem to be able to resolve some inconsistencies of EUT with empirical and experimental evidence. The disadvantage of the CPT is its complexity, which makes it difficult to apply in practice.

The next section presents the modern portfolio theory, which is based on the assumption that people choose among risky alternative according to the MVT, i.e. all we need to compare lotteries or financial assets is the mean and variance of their returns.

6. Mean Variance Portfolio Analysis and Capital Asset Pricing Model (CAPM)

We begin our analysis with simplifying assumptions. There are only two periods, and preferences over risky returns are represented with Mean-Variance Theory (MVT).That is, we assume that the mean-variance utility function is *v* (μ ,σ), where μ is the mean return and $\sigma^2$is the variance of the return.

6.1. Mean-Variance Portfolio Analysis

Suppose that there are n assets indexed by *i* = 1, 2 …n. The price of an asset *i* in the first period is q_i and in the second period the asset pays dividend D_i' and has the price q_i'. Thus, the total value of the asset in the second period is $A_i' = D_i' + q_i'$. The gross return on the asset is $R_i = A_i'/q_i$, and the net return is $r_i = R_i - 1$.

In our analysis we focus primarily on the net returns, since most of the examples we will encounter present data on net returns. From the point of view of the investor, who makes portfolio decisions in the first period, the return on a given asset, r_i, is a random variable, with mean (expected value) and variance.

Let [Mean] : $\mu_i = E(r_i)$

[Variance]: $\sigma_i^2 = Var(r_i)$

And [Standard deviation] : $\sigma_i = \sqrt{Var(r_i)}$

The standard deviation (or variance) tells us how volatile the asset returns are. An asset that guarantees a particular return r_f with certainty, is called a risk-free asset, and we have $E(r_f) = r_f$, and $Var(r_f) = 0$.

For any two assets, *i* and *j*, we will also be interested in the degree of comovement of their returns (r_i and r_f), which is measured with covariance or correlation: [Covariance] : $\sigma_{ij} = Cov(r_i, r_f)$ or [Correlation]

: $\rho_{ij} = \frac{Cov(r_i, r_f)}{\sigma_i \sigma_j}$.

We can see almost terms and formulas we use in this survey are from statistic. The mean and variance (or standard deviation) is the only two features of any asset (or portfolio of assets) that investors care about. Thus, any asset can be represented as a point in the mean-variance space, as figure illustrates.

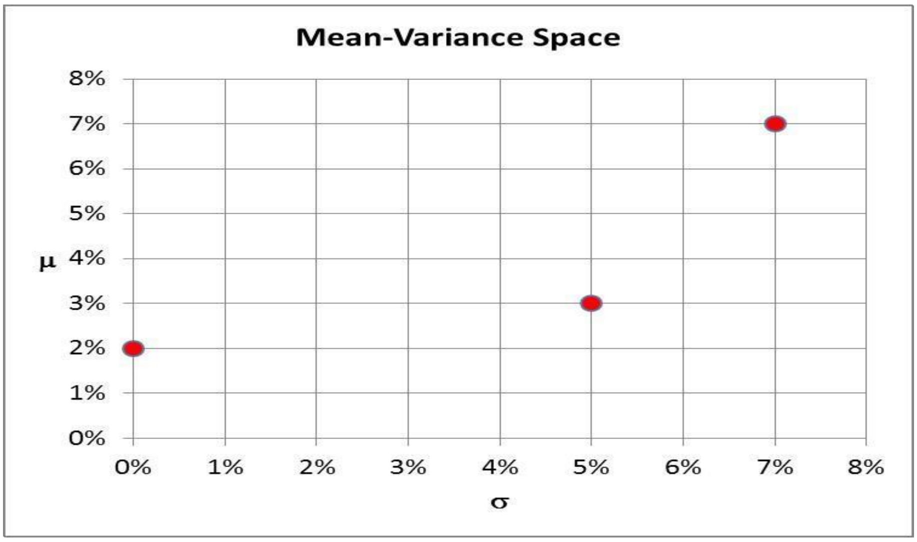


Figure 4: Mean-variance space.

The figure shows 4 assets: a risk-free asset with mean return of 2% and variance of zero, a risky asset with mean return of 3% and standard deviation of 5%, and another risky asset with mean return of 7% and standard deviation of 7%.The mean returns, variances and covariances are usually estimated based on historical data. Although figure 4 is called the mean-variance space, we prefer to use σ (standard deviation) on the *x*-axis instead of σ^2(variance), because standard deviations have the same units as the data, while the units of variance are the original units squared.

6.1.1. Portfolios with two risky assets

Suppose an investor divides some amount of his wealth, *w*, between two assets *i* and *j*, such that a fraction $\lambda \epsilon$ [0; 1] is invested in *i* and the rest $1-\lambda$ is invested in *j*. The net return on a portfolio is: $r_\lambda = \lambda\, r_i + (1-\lambda)\, r_j$

Since the returns $r_i$and r_j are random variables, the return on the portfolio r_λ is also a random variable. Thus, any portfolio composed of risky assets can be viewed as just another risky asset, with random return r_λ, and with mean and variance as follows:

$$\mu_\lambda = E(\lambda\, r_i + (1-\lambda)\, r_j) = \lambda\mu_i + (1-\lambda)\,\mu_j$$

$$\sigma_\lambda^2 = Var(\lambda\, r_i + (1-\lambda)\, r_j) = \lambda^2\sigma_i^2 + (1-\lambda)^2\sigma_j^2 + 2\lambda(1-\lambda)\rho_{ij}\sigma_i\sigma_j$$

$$\sigma_\lambda = \sqrt{[Var(\lambda\, r_i + (1-\lambda)\, r_j)]}$$

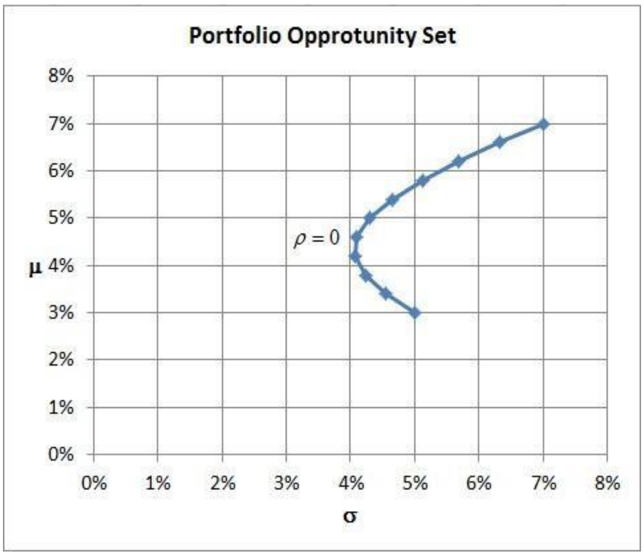
The mean return on the portfolio depends on the mean returns of the com-posing assets (μ_i and μ_j), as well as the asset shares λ and $(1-\lambda)$. Thus, the mean return is a weighted average of the mean returns on individual assets. The variance of the portfolio return depends on the variances of returns of the composing assets σ_i^2 and σ_j^2 on the asset shares λ and $1-\lambda$ and on the covariance (or correlation) of the asset returns.

Therefore, even an investor who cares only about mean and variance of his portfolio (MVT investor), will still need information about the correlations of assets in his portfolio.

We assume that asset returns are exogenous to investors, and investors can only choose the portfolio share (or portfolio weight) of each asset, $\lambda \epsilon$ [0; 1], (and $1-\lambda$) in this case. By varying $\lambda \epsilon$ [0; 1], the investors can create infinitely many portfolios, even if there are only two assets available. The set of all possible portfolios can be presented graphically, in the mean-variance space, as the mean-variance opportunity set. For any two assets with returns r_i and r_f , the mean-variance opportunity set, is the set of all mean and standard deviations of returns on portfolios created from investing a share $\lambda \epsilon$ [0; 1]in asset *i* and share $1-\lambda$ in asset *j*. Mathematically, the mean-variance opportunity set is defined as follows:

$$OS_{ij} = \left\{(\mu_\lambda, \sigma_\lambda) \epsilon R \times R_+ / \lambda \epsilon [0;\, 1],\ \mu_\lambda = \lambda\mu_i + (1-\lambda)\,\mu_j\,,\ \sigma_\lambda = \sqrt{[Var(\lambda\, r_i + (1-\lambda)\, r_j)]}\right\}$$

	Asset i	*Asset j*
μ	*3%*	*7%*
σ	*5%*	*7%*
ρ_{ij}	*0*	



Two assets

Figure 5: Mean-variance opportunity set for two assets.

Each point on the graph represents a portfolio created with a particular λ.

For example, the point *(3%, 5%)* is created from $\lambda = 1$, which means that the portfolio consists entirely of asset *i*. Similarly, the point *(7%, 7%)* is the portfolio corresponding to $\lambda = 0$, which means that the entire investment is in asset *j*. All the other points on the graph represent portfolios with $0 < \lambda < 1$, i.e. portfolios containing positive shares of both assets. What we need to know here is the standard deviation ("risk") of asset *i* is 5%, and when combined with a more risky asset j, with standard deviation of 7%, we are able to create a portfolio with around 4% standard deviation! This is a stunning feature, which is called the diversification effect. This terms means the reduction in portfolio risk (variance) that results from combining assets with certain statistical (probabilistic) features. For example:

"You are invested equally in a company that produces suntan lotion and a company that produces umbrellas. If the summer turns out to be sunny, the first company does well and the second poorly. In contrast, if the summer turns out to be rainy, the first company does poorly and the second does well. In other words, their returns are negatively correlated. By investing in both of them, instead of only one of them, obviously you reduce your portfolio risk a lot. Let see what happen with n risky assets.

6.1.2. Portfolios with n risky assets

Suppose there are n > 2 risky assets with random returns $r_1, r_2, \dots . r_n$ and $\lambda_1, \lambda_2, \dots . \lambda_n$ the portfolio shares (weights) of some fund allocated to these assets. The shares must add up to 1, which is like a "budget constraint": $\sum_{i=1}^{n} \lambda_i$

The return on a portfolio with given shares is $\lambda_1, \lambda_2, \dots . \lambda_n$

$$r_\lambda = \lambda_1 r_1 + \lambda_2 r_2 + \cdots . + \lambda_n r_n = \sum_{i=1}^{n} \lambda_i r_i$$

As before, the return on a portfolio is random, and has mean and variance

$$\mu_\lambda = \lambda_1\mu_1 + \lambda_2\mu_2 + \cdots . + \lambda_n \mu_n = \sum_{i=1}^{n} \lambda_i \mu_n$$

$$\sigma_\lambda^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i\lambda_j Cov(r_i, r_j)$$

Recall that variance of a random variable can be defined as the covariance of that random variable with itself. Therefore, the double summation terms for $i = j$ captures the individual variance terms of every returnr_i, multiplied by λ_i^2.

The notation becomes much easier if we express the above in matrix form. Moreover, matrix notation makes programming with Mat lab straightforward. The means of all asset returns can be expressed with the *n*-by-*1* vector μ

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix}_{n\times 1}$$

The covariance matrix of all individual assets is n-by-n matrix $\sum$:

$$\sum = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{bmatrix}_{n\times n}$$

Thus, the *ij*th element is $\sigma_{ij} = Cov(r_i, r_j)$, and the elements on the diagonal are the individual variances.

We introduce two more notations: the vector of portfolio weights and a vector of 1-s:

$$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}_{n\times 1}, \quad {}_{n\times 1} \quad I_n = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

With these notations, we can write the budget constraint $[BC]$ as:

$[BC]: \lambda' 1_n = 1$

The portfolio mean is: $\mu_\lambda = \lambda'\mu$ And the portfolio variance as: $\sigma_\lambda^2 = \lambda' \sum \lambda$

With more than 2 assets, the mean-variance opportunity set is not a curve, we have seen that combining two assets in a portfolio, creates an opportunity set which is a curve. Even with 3 assets, one can create portfolios with pairs of assets, and that will give three curves. But in addition to pairs, any portfolio consisting of two assets can be considered as another asset, which can be combined with individual assets in yet new portfolios. This is why the opportunity set is connected, i.e. does not have "holes".

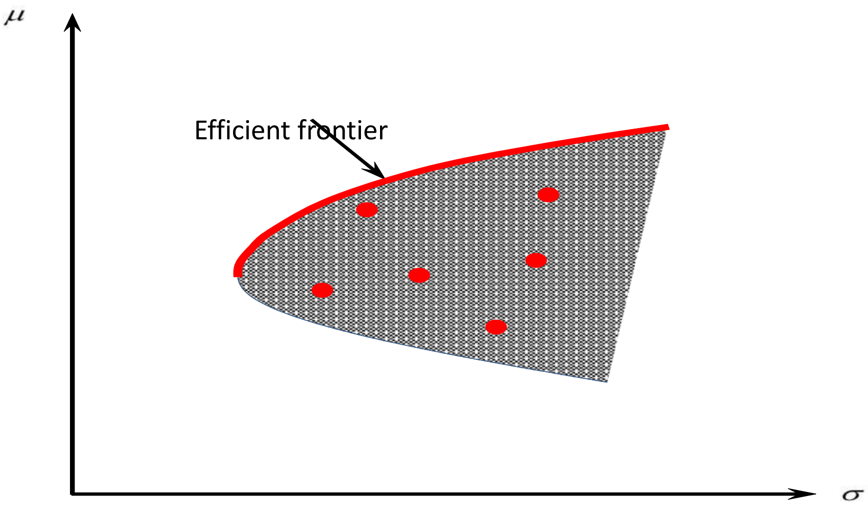


Figure 6: Mean-variance opportunity set, n > 2 assets.

Imagine that you choose portfolios that give you minimum variance of return for any level of mean return. These portfolios would be on the left boundary of the opportunity set, and are called the minimum-variance frontier. However, only the increasing part of the minimum-variance frontier constitutes the efficient frontier, because any portfolio below thee efficient frontier is dominated by some portfolio on the efficient frontier.

In other words, for any portfolio under the efficient frontier, there is another portfolio on the efficient frontier which has higher return and lower variance. Any investor should choose portfolios on the efficient frontier only.

To summarize, in this section we illustrated the shape of the mean-variance opportunity set, and concluded that any MVT investor, with monotone and variance averse utility function, will choose a portfolio on the efficient frontier only. However, we will show in the next section, that investors can in general do much better than the efficient frontier, if there exists a risk-free asset. In reality, there are such assets, for example government bonds or treasury bills that have a guaranteed return over one period.

6.1.3. Adding a risk-free asset

Suppose that in addition to the n risky assets with random returns $r_1, r_2, \ldots . r_n$, we also have a risk-free asset with guaranteed return r_f.

Let the portfolio share in the risk-free asset be λ_0, and the shares in other risky assets be as before $\lambda = [\lambda_1, \lambda_2, \ldots . \lambda_n]'$. The "budget constraint" on the weights is $\lambda_0 + \sum_{i=1}^{n} \lambda_i =$

1

Notice that the sum of weights on risky assets is $\sum_{i=1}^{n} \lambda_i = 1 - \lambda_0$. Let see an example:

The portfolio $\lambda_0 = 0.4$, $\lambda = [0.1, 0.2, \ 0.3]'$ consists of 40% investment in risk-free asset, and 10%, 20% and 30% investment in risky assets 1, 2 and 3. Thus, 60% of the portfolio is invested in risky assets. If we refer to the portfolio of the risky assets as a separate portfolio, then the weights have to add up to 100%, and this is achieved by dividing λ by the sum of its weights:

$$\frac{\lambda}{\sum_{i=1}^{n}\lambda_i} = \frac{\lambda}{1-\lambda_0} = \left[\frac{0.1}{0.6}, \frac{0.2}{0.6}, \frac{0.3}{0.6}\right]'$$

Consider a portfolio which combines a fraction λ_0 invested in the risk-free asset with $1-\lambda_0$ invested in some portfolio *p* consisting of the other n risky assets only. Notice that if $\lambda_0 < 0$, the investor borrows money at the risk-free return. The return on this new portfolio is:

$$r = \lambda_0 r_f + (1-\lambda_0) r_p$$

With mean and variance:

$$\mu_r = \lambda_0 r_f + (1-\lambda_0)\mu_p \quad \text{And} \quad \sigma_r^2 = (1-\lambda_0)^2\sigma_p^2, \quad \sigma_r = (1-\lambda_0)\sigma_p$$

Thus, combinations of the risk-free asset with any other portfolio of risky assets are located on the line that connects the risk free asset and this other portfolio *p*.

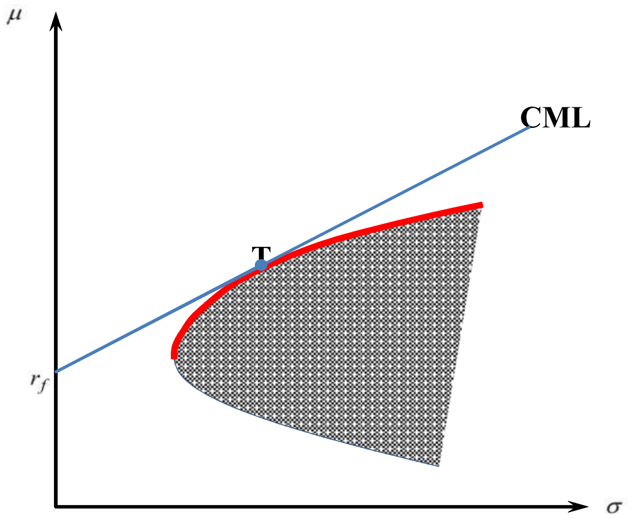


Figure 5: Capital Market Line (CML)

We see on figure 5 that the best investment opportunities are created when we combine the risk-free asset with the tangent portfolio on the efficiency frontier (portfolio T).

The line that connects the risk-free portfolio with the tangent portfolio is called the

Capital Market Line (CML). The slope of the capital market line is called the Sharpe ratio (SR), and is equal

$$SR = \frac{\mu_T - r_f}{\sigma_T}$$

The Sharpe ratio gives the tradeoff between risk and return, i.e. the excess return that investors can get over the risk-free asset, for every additional unit of risk (the standard deviation). For example, suppose that μ_T = 7%, r_f = 2%, σ_T = 10%. Then, the Sharpe ratio is

$$SR = \frac{\mu_T - r_f}{\sigma_T} = \frac{7-2}{10} = 0.5$$

, This means that an increase in risk of a portfolio (standard deviation) by 1% is compensated with a 0.5% higher expected return.

Notice that with the risk-free asset, investors have better set of portfolios to choose from (CML) than the efficient frontier, because the CML lies above the efficient frontier, and coincides with the efficient frontier only at the tangent portfolio. All investors will therefore choose some portfolios on the CML.

Those who desire less risk, will invest a greater proportion of their wealth in the riskfree asset (λ_0 is greater). These conservative investors will choose portfolios on the CML that are close to r_f. In other words, all investors should choose portfolios on the Capital Market Line.

In order to make optimal investment choices, we now need to find the tangent portfolio (which is the same as finding the Capital Market Line).

As you can see, finding the optimal portfolios analytically (the CML) is a tedious task, even with only two risky assets. With more than 3 risky assets, it is a mission impossible, and requires the use of computer optimization software.

6.2. Capital Asset Pricing Model (CAPM)

In the previous section we derived optimal portfolios for investors whose preferences are described by Mean-Variance Theory (MVT). Without a risk-free asset, we showed that investors will choose portfolios from the efficient frontier of the opportunity set.

Under the assumption that there is a risk-free asset, and that investors can borrow and lend at the risk-free rate r_f. We showed that all MVT investors will choose portfolios from the Capital Market Line (CML) which is the highest slope line connecting the riskfree asset with the mean-variance opportunity set. Thus, the CML is the set of optimal portfolios such that any investor with MVT preferences will choose from. The implication is that all investors hold the same portfolio of risky assets, called the tangent portfolio T, and the only difference between investors is the fraction $\lambda_0$invested in riskfree asset versus the fraction invested in the tangent portfolio T.

Since all investors hold the same portfolio of risky assets, shares of risky assets $\lambda_1, \lambda_2, \dots . \lambda_n$ in the tangent portfolio are common to all investors. For example, if IBM stock represents 0.1% of investor A portfolio of risky assets, it also represents 0.1% of investor B portfolio of risky assets. Suppose there are I investors. The market share of asset i, out of the total market value of all risky assets is:

$$\frac{\sum_{j=1}^{I} \lambda_i \, \omega_i}{\sum_{j}^{I} \sum_{i=1}^{n} \lambda_i \, \omega_i} = \frac{\omega_i \sum_{j=1}^{I} \lambda_i}{\sum_{j=1}^{I} \omega_i \sum_{i=1}^{n} \lambda_i} = \frac{\lambda_i W}{W} = \lambda_i$$

Where ω_i is investor j wealth (money) invested in the market for risky assets (say the stock market), and $\sum_{j=1}^{I} \omega_i$= W is the total value of all the risky-assets. Therefore, since all the investors hold all the risky assets, the individual share of asset i in the tangent portfolio is also the market share of asset *i*, and the tangent portfolio T in equilibrium is also the Market portfolio M. Thus, in this section we will refer to the tangent portfolio as the market portfolio.

6.2.1 Deriving the CAPM

The starting point is the result that all investors hold the same market portfolio M of risky assets (which is the same as the tangent portfolio T). The market portfolio contains positive shares of all existing assets.

Next we examine the effect of slightly changing the share of some security *i*. Consider mixing a small fraction ω of some security *i* with $1 - \omega$ of the market portfolio.

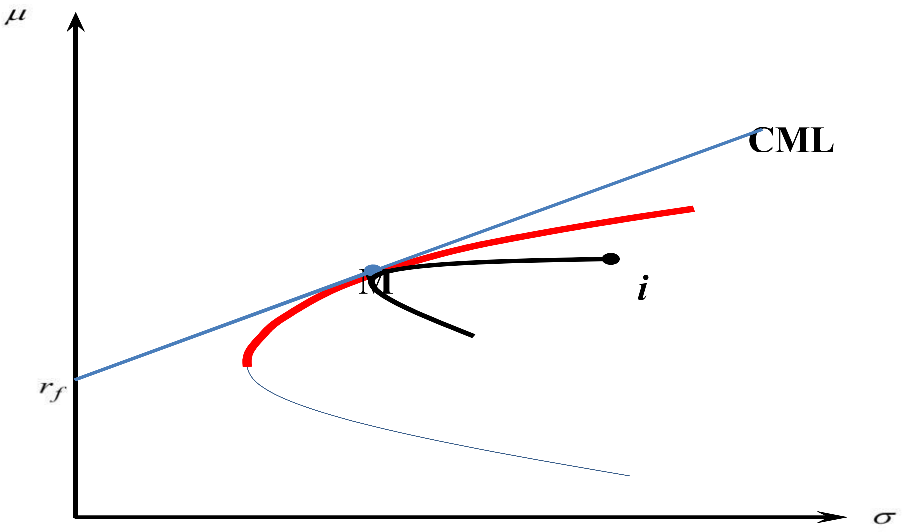


Figure 6: Market portfolio mixed with asset i.

When $\omega = 0$the *i*-curve coincides with the market portfolio. For$0 < \omega \leq 1$, the combination is between M and *i*. Values of $\omega < 0$ mean that we are selling some of the existing holdings of asset *i* in the market portfolio.

The return on portfolios that combineω of asset *i* and$1 - \omega$ of the market portfolio is:

$r_p = \omega r_i + (1 - \omega)r_M$

With mean and variance:

$\mu_p = \omega\mu_i + (1 - \omega)\mu_M$ And $\sigma_p^2 = \omega^2\sigma_i^2 + (1 - \omega)^2\sigma_M^2 + 2\omega(1 - \omega)Cov(r_i, r_M)$

6.2.2. CAPM in practice

We studied optimal portfolio selection under the assumption that all investors care about is the mean and variance of the return to their investment. In other words, we assumed that preferences are described by the Mean-Variance Theory (MVT). This is not the same as assuming that there is only one investor, or that all investors are identical.

In our examples we assumed that the mean-variance utility of individual *i* is:

$\mu^i(\mu, \sigma) = \mu - \frac{\gamma_i}{2}\sigma^2$

Thus, there could be unlimited number of investors, which differ by their varianceaversion (or risk aversion) parameterγ_i. Despite the differences in preferences among individual investors, we arrived at a striking conclusion - the Two-Fund Separation Theorem, which implies that all investors will invest in two funds only: the risk-free asset and market portfolio. The only difference between investors' portfolios is the fraction of their financial wealth invested in the risk-free asset,λ_0, more variance-averse investors will hold a larger portion of their financial wealth in the risk-free asset. But nevertheless, all investors will hold the same market portfolio of risky assets - M.

Moreover, we showed that all assets (and all portfolios for that matter), must lie on the same Security Market Line. As follow this equation: $\mu_i - r_f = \beta_i(\mu_M - r_f)$

The expected excess return on asset *i* (or any portfolio) is proportional to its Beta:

$$\beta_i = \frac{Cov(r_i, r_M)}{Var(r_M)} = \rho_{iM}\frac{\sigma_i}{\sigma_M}$$

The term $\mu_i - r_f$ is the expected excess returns on asset *i*, and $\beta_i(\mu_M - r_f)$ is the predicted excess return by the CAMP (SML). Asset i has mean return higher than the one predicted by the SML (positive Alpha), while asset j has mean return lower than the predicted by SML (negative Alpha).

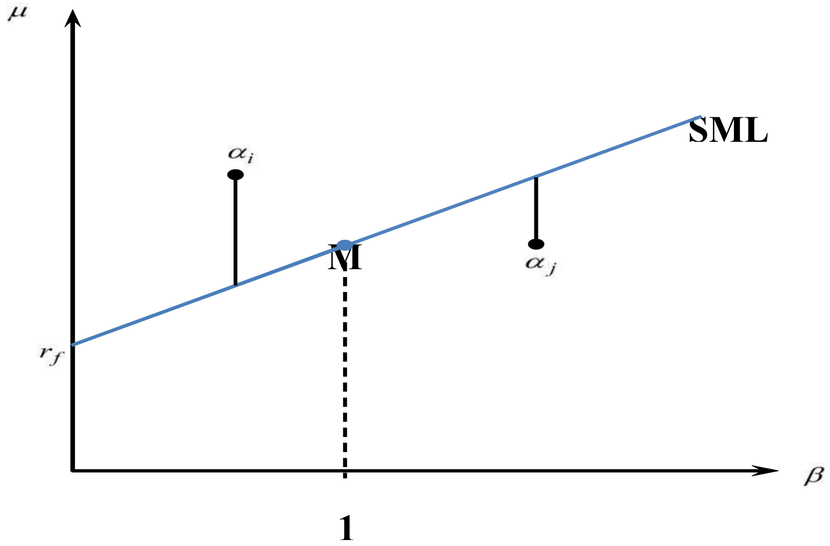


Figure 7: The Alpha of an asset (or portfolio).

Our first instinct tells us that positive Alpha of an asset (or portfolio) is an indicator that the asset outperformed the market, and had average return above what the theory (CAPM) predicts.

Similarly, a negative Alpha indicates that an asset (or portfolio) underperforming, and delivering returns below the levels required by its Beta risk. A very appealing (but dangerous!) investment strategy of buying assets with positive Alpha and short selling assets with negative Alpha.

One explanation for observing non-zero Alphas is that different investors tend to specialize in different subsets of assets. If a hedge fund specializes in particular industries, its tangent portfolio will consist of that industry's securities.

The derivation of the CAPM shows that if the tangent portfolio consists of a group of assets, then all the assets that make up the tangent portfolio, or any combination of these assets, must have Alpha of zero.

The practical implication of this is that all investors need to choose whether to actively manage their portfolio (be active investors) or be passive. Active investors will choose their own tangent portfolio, while passive investors will simply invest in some mutual fund (index fund). One needs to remember that μ and $\sum$ are not given, and active investors must estimate the mean returns and covariance's based on historical data, and sometimes make predictions based on firm-specific factors, industry-specific factors and macroeconomic forecasts. Thus, being active investor entails a cost, and it can be shown theoretically that only the most efficient and informed investors will be active in equilibrium.

7. Conclusion

This paper presents the theory of decision under uncertainty which develop alternative theories for choices under risk. To summarize, the prospect theory (PT or CPT) is more general than the EUT, and seem to be able to resolve some inconsistencies of EUT with empirical and experimental evidence. The disadvantage of the CPT is its complexity, which makes it difficult to apply in practice. Of the three theories, the CPT is by far the most complicated, while the MVT is the simplest.

The result show the modern portfolio theory, which is based on the assumption that people choose among risky alternative according to the MVT. All we need to compare lotteries or financial assets is the mean and variance of their returns.

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