Prime numbers demystified

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Introduction

There are many prime numbers algorithms that man has devised to predict the next prime number. The common denominator, however, is that none of them can predict all prime numbers to infinite with one hundred per cent accuracy. Prime numbers can best be described as semi-random or partly random because of the many bits and pieces of arithmetic series patterns that describe them. The reason why this author is writing this paper is so that he can share with you what might be the ultimate prime numbers algorithm that decisively explains why the prime numbers series are as incoherent as they are. In other words this algorithm offers simple explanations to many petty questions about prime numbers that would otherwise require complicated solutions.

1.1 How to calculate a prime number

One of the fastest methods of calculating prime numbers is the **Eratosthenes sieve**, named after Eratosthenes (276BC-194BC). Prime numbers are worked out by successively filtering out all multiples of $2 \leq x \leq \sqrt{n}$, where x are the prime factors of all non-prime numbers less than n, where n is the greatest whole number of the sieve. Table 1.1 is an example of such a sieve. Cancelling out multiples of 2, 3, 5 and 7 leaves only prime numbers less than 100.

1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

Table1.1

Even though prime numbers cannot form a coherent, regular pattern, they can be arranged to make quite beautiful spirals. An example is the Ulam spiral (after Stanislaw Ulam) and many other types of spirals that can be readily viewed on ***mathworld.wolfram.com/PrimeSpiral.html*** .It is said that odd numbers cannot make spirals of patterns like prime numbers. But randomly picking odd numbers to try to form a spiral is **a mathematical error** since prime numbers generally exist as **two distinct sets** and members of each set always have a difference that is a multiple of six. That means prime numbers do not happen entirely randomly along the number line but follow a certain rule and that would be discussed in due course.

1.2 Pattern of remainders

There are quite a huge number of arithmetic sequences that are identifiable in prime numbers. For example 2n+1, where n is a whole number, is a universal set that certainly includes all prime numbers. 6n∓1 is another set of integers that definitely includes all prime numbers greater than or equal to 5. The list of such polynomials is endless. For further reading see, Green-Tao theorem: An exposition, by David Conlon, Jacob Fox, and Yufei Zhao. The purpose of the next two sections is to explain in simple terms why there is an infinite number of disjointed prime numbers arithmetic series.

	2	3			
1	1				
2	0				
3	1				
4	0				
5	1				
6	0				
7	1				
8	0				
9	1	0			
10	0	1			
11	1	2			
12	0	0			
13	1	1			
14	0	2			
15	1	0			
16	0	1			
17	1	2			
18	0	0			
19	1	1			
20	0	2			
21	1	0			
22	0	1			
23	1	2			
24	0	0			
25	1	1	0		
26	0	2	1		

Table 1.2

The above pattern is a tabular expression of equation $2 \leq x \leq \sqrt{n}$. To obtain the column under the bold green 2, you divide each integer by 2, write only the remainder, and ignore the rest of the quotient. The second column of remainders is obtained in the same way; you divide the integer by 3 (see the green bold 3 at the top of the column). Therefore the row of remainders adjacent to, say 11, is obtained by first dividing 11 by 2. The remainder is 1, which is the 1 adjacent to 11. The 2 adjacent to 11 in the second column of the remainders is obtained by dividing 11 by 3, the remainder is 2.

The method used above is the one used to obtain the entire pattern from 1 to 26 or the longer one at the end of this paper. The reader should also note that none-prime integers all have a row of remainders in front of them that have at least one zero, for example 20, thus indicating that they are not prime. All prime numbers greater than 9 have rows of remainders that are all devoid of zeros, e.g. 13, 17 19 etc. If the column of remainders under each bold green prime is read downwards, the reader would notice that the pattern of the remainders repeats after every *nth* interval (where *n* is the bold green prime number on top of each respective column). For example, the column of remainders under $n=5$, which begins at 25, reads; 0, 1, 2, 3 4, 0, 1, and so on to infinite (see longer pattern at the end of the paper).

Some mathematicians would know straight away why the rows of remainders adjacent to 11 and 13(or any other row of remainders adjacent to any integer) repeat after every sixth integer right to infinite. Under the column with a bold two (see table 1.2) there are only two possible remainders; 1 and 0. Under the second column with a bold 3 there are only three possible remainders; 0, 1and 2. Therefore there are 2×3 ways of forming unique rows of remainders from integers 9 to 24 of table 1.2. Therefore the "prime" rows of remainders adjacent to 11 and 13 repeat after every sixth integer right up to infinite(check the pattern at the end of the paper) and that is the phenomenon that is also responsible for **twin primes** since 11 and 13 are prime numbers next to each other. The two types of prime numbers have been highlighted in red and blue so that the reader can see the so-called "**prime numbers race**". The rows of remainders adjacent to other integers are cyclic as well.

Since the rows of remainders repeat after every sixth integer we can use that pattern to predict the next prime integer in each set of prime numbers between 8 and 25. Indeed 11+6, 11+6+6 are prime numbers. The method can be used to predict the primes in the other set as well; 13+6 is prime nevertheless 13+6+6 is not prime since equation $2 \leq x \leq \sqrt{n}$ (where *x is* 2 and3) applies to prime numbers between 8 and 25. After 25 the method of adding intervals is no longer 100% reliable and that would be explained in due course nevertheless all prime numbers of each set would always have a difference that is a multiple of six. The repetition of rows of remainders is also noticeable in other integers. For example all multiples of 6 have a repeating row of remainders 00. The row of remainders (02) adjacent to 8 repeats after every sixth integer from 8 to infinite as you can see at the table at the end of this paper. If we look at the table at the long table we see that the rows of remainders exist in six main sets (S_1, S_2, S_3, S_4, S_5, S_6);

S_1	0 0…	6, 12, 18⋯∞
S_2	1 1…	7, 13, 19⋯∞
S_3	0 2…	8, 14, 20⋯∞
S_4	1 0…	9, 15, 21⋯∞
S_5	0 1…	10, 16, 22⋯∞
S_6	1 2…	11, 17, 23⋯∞

Table 1.3

In the table above, we have divided **all whole numbers** into six sets; set S_1, S_2, S_3, S_4, S_5 and S_6. The middle column of the table indicates the first two remainders of the repeating row adjacent to every element of each set. For example the rows of remainders of set S_2 {7, 13, 19…} all begin with 1, 1. The third column is the list of the elements (integers) of each set. We did not include whole numbers from 0 to 5 but they too can be manually included in these sets as well even though they have different rows of remainders. Whole number 0 can be manually inserted into set S_1, since 0+6=6. Whole number 1 belongs to S_2, since 1+6 =7 and so on. We are trying to demystify prime numbers so that the reader realises that prime numbers are just members of one of the many arithmetic series based on the repetition of the row of remainders after every sixth, thirtieth integer etc. Sets S_2 and S_6 are unique because there are arithmetic series patterns that **include** numbers whose rows of remainders are devoid of zeros, which we call prime numbers. Some members of these two sets are, however, not prime, for example 25, 35, 49, etc. Nevertheless the reader has been given a clear proof why prime numbers can form so many arithmetic series patterns. They belong to sets S_2 and S_6, which are themselves arithmetic series patterns.

1.3 Arithmetic series beyond integer 24

After 24 the pattern of remainders has to include three rows because the elements of the set $2 \leq x \leq \sqrt{n}$ are now 2, 3, and 5 and that is only valid between 24 and 49. Within that region of integers a similar row repeats after 30 (2×3×5) integers. If 29's row of reminders are of interest, such a row of remainders repeats at 29+30 and the result would be 59(59 is prime!). We can add 30 to both 31 and 37 and still obtain prime numbers even though we would be violating the equation $2 \leq x \leq \sqrt{n}$ since we used the interval 30 (which applies to integers from 25 to 48 only). After 48 the interval to be used to predict the next prime number becomes 210(2 x 3 x 5 x 7). That interval is only relevant from 49 to 120. Nevertheless 59+210; 61+210; 67 +210 and 73+ 210 are all prime (check the pattern at the end of the paper). Therefore we realize that each row of remainders does indeed repeat after a certain interval that is always a multiple of 6 or of the previous intervals but it would be having extra columns of remainders because these patterns of remainders for integers greater than 24 always overlap into other regions of integers where they do not apply. For example the rows of remainders adjacent to 29 or 31 repeat after every 30th integer up to infinite but wherever they repeat they would be in a region of integers that have extra columns of remainders. The overlapping of intervals into intervals of the **next prime squares** makes it impossible to predict prime numbers greater than 24 with 100% certainty. Nevertheless each row of remainders adjacent to any integer repeats after $a \times b \times c \ldots n$, where *a, b, c,* up to *n* are the green bold integers at the top of the remainders column (the elements of set $2 \leq x \leq \sqrt{n}$). Wherever a new column of remainders is introduced, **each arithmetic progression subdivides or branches into even more sets.**

As an example, from 9 to 24 the rows of remainders adjacent to prime numbers are only one-one (1, 1) and one-two (1, 2) see table 1.2 and table 1.3. From 25 to 48, the one-one row of remainders subdivides into one-one-one(1,1,1), one-one-two(1,1,2) and one-one-three(1,1,3) rows of remainders. The one-two (1, 2) row of remainders beginning at integer 11 also subdivides as can be seen at the pattern of remainders at the end of the paper.

However, all the remainders repeat after an interval that is a multiple of six due to the fact that the first two elements of the set $2 \leq x \leq \sqrt{n}$ are always 2 and 3. In fact, new sub-arithmetic series patterns begin at every prime square of the pattern of remainders; the common difference being the product of the prime numbers in the set$2 \leq x \leq \sqrt{n}$. Nevertheless, that is unimportant and cannot enable us to predict prime numbers greater than 24 with total accuracy but it enables us to **easily** explain why twin primes, and some other semi-regular patterns are found in prime numbers right to infinite.

1.4 Are prime numbers infinite?

Euclid (about 325BC -265BC) proved that there are an infinite number of prime numbers (Martin H. Weissman, why prime numbers still fascinate mathematicians 2,300 years later, April 2, 2018 6.47 a.m. EDT). We want to use the table/pattern of remainders at the end of this paper to show that there are an infinite number of prime numbers(let the reader refer to the table at the end of the paper). Suppose that 11 is the greatest prime number such that all other numbers greater than it are composite. Therefore that means there would not be a need to introduce the sixth column of remainders adjacent to 169 (if 11 is the greatest prime then 13 is not a prime number hence169 cannot be a prime square). Therefore that means 169 itself would be prime! Furthermore all those prime numbers rows of remainders occurring between 121 to 168 would continue to repeat at regular intervals to infinite thus "creating" more primes! In fact the absence of the sixth row adjacent to 169 would enable a prediction of prime numbers from 169 to infinite with one-hundred per cent certainty, since only the same rows of remainders would be recurring without a need to add extra columns of remainders!

Thus a termination of prime numbers along the number line causes the square of that greatest prime number to be prime and causes non-primes like 169 and 221 to be prime! Such a phenomenon creates even more prime numbers along the number line. Therefore a cessation of prime numbers along the number line means the square of the last prime number would be prime and the way these rows of remainders repeat means an occurrence of more prime numbers beyond the "last prime"! So the very cessation of prime numbers along the number line makes the square of the largest prime number prime. That is an impossibility since a square cannot be prime. Therefore the above statement is a proof that prime numbers exist right to infinite.

NB. This above proof is only valid assuming that the algorithm used at the end of this paper to calculate or identify prime numbers is as natural and obvious or solid as the whole number line itself.

1.5 Linear prime numbers graphs

In section 1.3 we have shown you that prime numbers can be divided into two main sets; the set whose initial prime number is 11 while the other set is a set whose smallest integer is 13. As we showed earlier, the former set has its row of remainders that always begins with 1 and 2. The other set has a row that always begins with 1 and 1. That is why we call all prime numbers of the set S_6, that includes 11, as *one-two* prime numbers. The other set S_2, is called the *one-one* prime numbers set.

If you look carefully at the prime numbers at the end of this paper, you will realise that one-two and one-one prime numbers **are just terms** of an **arithmetic series** that begin at 11 and 13 respectively (all other integers are also elements of their unique arithmetic series patterns as well that continually subdivide at every prime square). The general term a_n describing the one-two arithmetic series is

$a_{1,2} = 11 + 6x, where\ x\ is\ a\ whole\ number, and\ the\ terms\ being$

11, 17, 23, 29$\cdots\infty$

The one-one arithmetic series general term is;

$a_{1,1} = 13 + 6x, where\ x\ is\ a\ whole\ number$, and the terms being 13, 19, 25, 31$\cdots\infty$

Looking at the above general terms of both equations, it becomes apparent that prime numbers are just one of the terms (maybe random) of a specific arithmetic series described by the above equations.

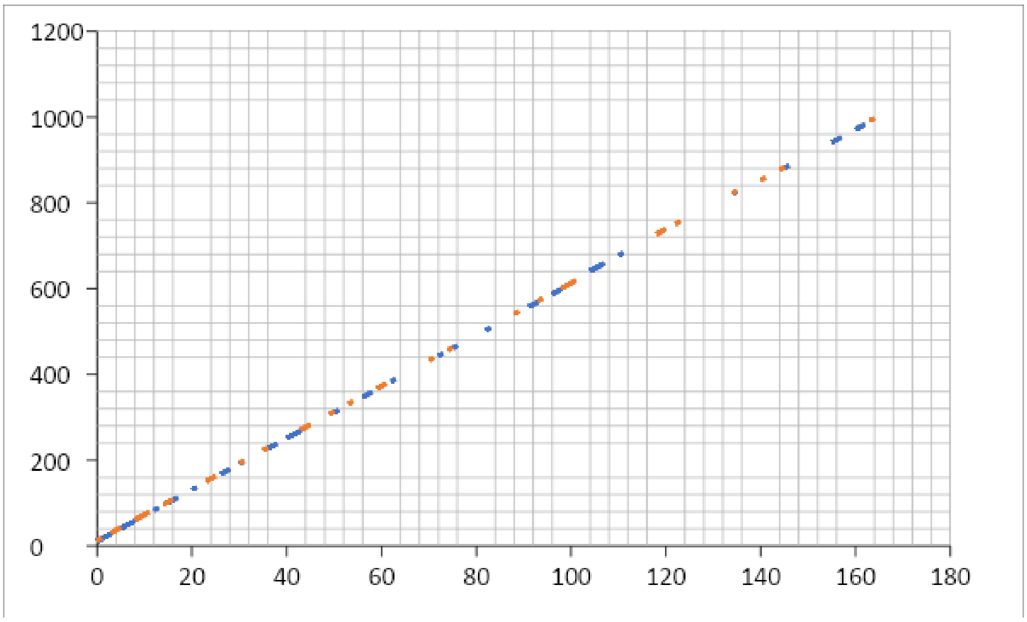


Fig1.1

The graph shown above is of equations $a_{1,2} = 11 + 6x$ and $y_{1,1} = 13 + 6x$ respectively. It is a plot of prime numbers of both sets against x, check the table of values below. The non-prime numbers were deliberately omitted; only prime numbers were used. Due to the fact that the linear equations are too close, both graphs appear as a single line with gradient 6. Therefore whatever prime number greater than 9 that you can think of; it will lie on either line $a_{1,2} = 11 + 6x$ or $a_{1,1} = 13 + 6x$, even if the lines are extrapolated to infinite.

x	6x+11	6x+13	x	6x+11	6x+13	x	6x+11	6x+13	x	6x+11	6x+13	x	6x+11	6x+13	x	6x+11	6x+13
			29			59		367	89		547	118		727	149		907
0	11	13	30	191	193	60		373	90			119		733	150	911	
1	17	19	31	197	199	61		379	91	557		120		739	151		919
2	23		32			62	383		92	563		121	743		152		
3	29	31	33		211	63	389		93	569	571	122		751	153	929	
4		37	34			64		397	94		577	123		757	154		937
5	41	43	35		223	65	401		95			124	761		155	941	
6	47		36	227	229	66		409	96	587		125		769	156	947	
7	53		37	233		67			97	593		126	773		157	953	
8	59	61	38	239	241	68	419	421	98	599	601	127			158		
9		67	39			69			99		607	128		787	159		967
10	71	73	40	251		70	431	433	100		613	129			160	971	

11		79	41	257		71		439	101	617	619	130	797		161	977	
12	83		42	263		72	443		102			131			162	983	
13	89		43	269	271	73	449		103		631	132	809	811	163		991
14		97	44		277	74		457	104			133			164		997
15	101	103	45	281	283	75	461	463	104	641	643	134	821	823	165		
16	107	109	46			76	467		105	647		135	827	829	166		1009
17	113		47	293		77			106	653		136					
18			48			78	479		107	659	661	137	839				
19		127	49		307	79		487	108			139					
20	131		50	311	313	80	491		109		673	140		853			
21	137	139	51	317		81		499	110	677		141	857	859			
22			52			82	503		111	683		142	863				
23	149	151	53		331	83	509		112		691	143					
24		157	54		337	84			113			144		877			
25		163	55			85	521	523	114	701		145	881	883			
26	167		56	347	349	86			115		709	146	887				
27	173		57	353		87			116			147					
28	179	181	58	359		88		541	117	719		148					

Table 1.4

Equations $a_{1,2}$ *or* $a_{1,1}$ *can be generalised to* $y = 6x + c$, where c is either 11 or 13. Therefore, if you are moving along the number line of the whole numbers $\mathbb{Z}_0$, and you are standing right on a prime number, whether a *one-one or one-two*, you could use the above equation to guess the position of the next prime number of either set. For example, assume that you are standing on prime number 17 and you want to guess both the next one-one prime and the one-two prime. Knowing that 17 is a one-two prime, you would guess the next prime by adding 6. Since the corresponding terms of these two arithmetic series have a difference of two, you would only need to add 8(6+2) to estimate the next one-one prime, which unfortunately would be 25. Nevertheless if you add 6 to 17+6, you get 29. If you add 8(6+2) to 23 to guess the next one-one prime you get 31. Therefore the author assumes the reader realises again that prime numbers are just elements of an arithmetic series.

1.6 Modified prime numbers sieve

A typical prime numbers sieve includes all whole numbers up to a certain integer *n*. Since it is known that prime numbers are only terms of a certain arithmetic series patterns, there is no need to include every term in the sieve, check example below. Only the arithmetic series terms, $6x + 11$ and $6x + 13$ are included in the sieve.

NB

Integers from 1 to 8 have to be manually included though. Integers 1, 2 and 3 have a difference of 1. Their pattern of "prime numbers" breaks down at 4, where another pattern of prime numbers begins whose common difference is 2[5,7]. The pattern breaks down at 9, where two patterns of arithmetic series emerge with a common difference of six. Such a pattern breaks down at the next prime square 25. Nevertheless 1 is not considered a prime number.

1	2	3	4	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11		13				17		19	
		23		~~25~~				29	
31				~~35~~		37			
41		43				47		~~49~~	
		53		~~55~~				59	
61				~~65~~		67			
71		73				~~77~~		79	
		83		~~85~~				89	
~~91~~				~~95~~		97			~~100~~

Table 1.5

1.7 <u>Odd numbers Ulam spiral</u>

It is claimed that odd prime numbers cannot form a spiral (see many failed spirals on **Wolfram**). Based on what you have learnt so far, you know that it is a mathematical error to highlight all odd numbers and hope to get a spiral. The odd numbers to be highlighted are one-one and one-two odd numbers of the sets; $S_6 = \{6x + 11\}\ and\ S_2 = \{6x + 13\}$ (see table below).

If we then highlight the odd numbers of the mentioned sets we obtain a pattern similar to that obtained with prime numbers. See figure 1,2. Such a result confirms our theory that prime numbers are just elements of an ordinary arithmetic series pattern.

x	6x+11	6x+13	x	6x+11	6x+13	x	6x+11	6x+13	x	6x+11	6x+13	x	6x+11	6x+13	x	6x+11	6x+13
			29	185	187	59	365		89	545		118	725		149	905	
0			30			60	371		90	551	553	119	731		150		913
1			31			61	377		91		559	120	737		151	917	
2		25	32	203	205	62		385	92		565	121		745	152	923	925
3			33	209		63		391	93			122	749		153		931

			3			6						12			15		
4	35		4	215	217	4	395		94	575		3	755		4	935	
			3			6						12			15		
5			5	221		5		403	95	581	583	4		763	5		943
			3			6						12			15		
6		49	6			6	407		96		589	5	767		6		949
			3			6						12			15		
7		55	7		235	7	413	415	97		595	6		775	7		955
			3			6						12			15		
8			8			8			98			7	779	781	8	959	961
			3			6						12			15		
9	65		9	245	247	9	425	427	99	605		8	785		9	965	
1			4			7			10			12			16		
0			0		253	0			0	611		9	791	793	0		973
1			4			7			10			13			16		
1	77		1		259	1	437		1			0		799	1		979
1			4			7			10			13			16		
2		85	2		265	2		445	2	623	625	1	803	805	2		985
1			4			7			10			13			16		
3		91	3			3		451	3	629		2			3	989	
1			4			7			10			13			16		
4	95		4	275		4	455		4	635	637	3	815	817	4	995	
1			4			7			10			13			16		
5			5			5			4			4			5	1001	1003
1			4			7			10			13			16		
6			6	287	289	6		469	5		649	5			6	1007	
1			4			7			10			13					
7		115	7		295	7	473	475	6		655	6	833	835			
1			4			7			10			13					
8	119	121	8	299	301	8		481	7			7		841			
1			4			7			10			13					
9	125		9	305		9	485		8	665	667	9	845	847			
2			5			8			10			14					
0		133	0			0		493	9	671		0	851				
2			5			8			11			14					
1			1		319	1	497		0		679	1					
2			5			8			11			14					
2	143	145	2	323	325	2		505	1		685	2		865			
2			5			8			11			14					
3			3	329		3		511	2	689		3	869	871			
2			5			8			11			14					
4	155		4	335		4	515	517	3	695	697	4	875				
2			5			8			11			14					
5	161		5	341	343	5			4		703	5					
2			5			8			11			14					
6		169	6			6	527	529	5	707		6		889			
2			5			8			11			14					
7		175	7		355	7	533	535	6	713	715	7	893	895			
2			5			8			11			14					
8			8		361	8	539		7		721	8	899	901			

Table 1.6

925	924	923	922	921	920	919	918	917	916	915	914	913	912	911	910	909	908	907	906	905	904	903	902	901	900	899	898	897	896	895	
926	809	808	807	806	805	804	803	802	801	800	799	798	797	796	795	794	793	792	791	790	789	788	787	786	785	784	783	782	781	894	
927	810	701	700	699	698	697	696	695	694	693	692	691	690	689	688	687	686	685	684	683	682	681	680	679	678	677	676	675	780	893	
928	811	702	601	600	599	598	597	596	595	594	593	592	591	590	589	588	587	586	585	584	583	582	581	580	579	578	577	674	779	892	
929	812	703	602	509	508	507	506	505	504	503	502	501	500	499	498	497	496	495	494	493	492	491	490	489	488	487	576	673	778	891	
930	813	704	603	510	425	424	423	422	421	420	419	418	417	416	415	414	413	412	411	410	409	408	407	406	405	486	575	672	777	890	
931	814	705	604	511	426	349	348	347	346	345	344	343	342	341	340	339	338	337	336	335	334	333	332	331	404	485	574	671	776	889	
932	815	706	605	512	427	350	281	280	279	278	277	276	275	274	273	272	271	270	269	268	267	266	265	330	403	484	573	670	775	888	
933	816	707	606	513	428	351	282	221	220	219	218	217	216	215	214	213	212	211	210	209	208	207	264	329	402	483	572	669	774	887	
934	817	708	607	514	429	352	283	222	169	168	167	166	165	164	163	162	161	160	159	158	157	206	263	328	401	482	571	668	773	886	
935	818	709	608	515	430	353	284	223	170	125	124	123	122	121	120	119	118	117	116	115	156	205	262	327	400	481	570	667	772	885	
936	819	710	609	516	431	354	285	224	171	126	89	88	87	86	85	84	83	82	81	114	155	204	261	326	399	480	569	666	771	884	
937	820	711	610	517	432	355	286	225	172	127	90	61	60	59	58	57	56	55	80	113	154	203	260	325	398	479	568	665	770	883	
938	821	712	611	518	433	356	287	226	173	128	91	62	41	40	39	38	37	54	79	112	153	202	259	324	397	478	567	664	769	882	
939	822	713	612	519	434	357	288	227	174	129	92	63	42	29	28	27	36	53	78	111	152	201	258	323	396	477	566	663	768	881	
940	823	714	613	520	435	358	289	228	175	130	93	64	43	30	25	26	35	52	77	110	151	200	257	322	395	476	565	662	767	880	
941	824	715	614	521	436	359	290	229	176	131	94	65	44	31	32	33	34	51	76	109	150	199	256	321	394	475	564	661	766	879	
942	825	716	615	522	437	360	291	230	177	132	95	66	45	46	47	48	49	50	75	108	149	198	255	320	393	474	563	660	765	878	999
943	826	717	616	523	438	361	292	231	178	133	96	67	68	69	70	71	72	73	74	107	148	197	254	319	392	473	562	659	764	877	998
944	827	718	617	524	439	362	293	232	179	134	97	98	99	100	101	102	103	104	105	106	147	196	253	318	391	472	561	658	763	876	997
945	828	719	618	525	440	363	294	233	180	135	136	137	138	139	140	141	142	143	144	145	146	195	252	317	390	471	560	657	762	875	996
946	829	720	619	526	441	364	295	234	181	182	183	184	185	186	187	188	189	190	191	192	193	194	251	316	389	470	559	656	761	874	995
947	830	721	620	527	442	365	296	235	236	237	238	239	240	241	242	243	244	245	246	247	248	249	250	315	388	469	558	655	760	873	994
948	831	722	621	528	443	366	297	298	299	300	301	302	303	304	305	306	307	308	309	310	311	312	313	314	387	468	557	654	759	872	993
949	832	723	622	529	444	367	368	369	370	371	372	373	374	375	376	377	378	379	380	381	382	383	384	385	386	467	556	653	758	871	992
950	833	724	623	530	445	446	447	448	449	450	451	452	453	454	455	456	457	458	459	460	461	462	463	464	465	466	555	652	757	870	991
951	834	725	624	531	532	533	534	535	536	537	538	539	540	541	542	543	544	545	546	547	548	549	550	551	552	553	554	651	756	869	990
952	835	726	625	626	627	628	629	630	631	632	633	634	635	636	637	638	639	640	641	642	643	644	645	646	647	648	649	650	755	868	989
953	836	727	728	729	730	731	732	733	734	735	736	737	738	739	740	741	742	743	744	745	746	747	748	749	750	751	752	753	754	867	988
954	837	838	839	840	841	842	843	844	845	846	847	848	849	850	851	852	853	854	855	856	857	858	859	860	861	862	863	864	865	866	987
955	956	957	958	959	960	961	962	963	964	965	966	967	968	969	970	971	972	973	974	975	976	977	978	979	980	981	982	983	984	985	986

Fig.1.2

The reader can clearly see the regular patterns obtained in the spiral above. The spiral begins at integer 25. The reason why spirals of odd numbers usually fail is because the authors of such spirals fail to realise that prime numbers are elements of an arithmetic series and odd numbers to be used to form such spirals should be elements of an arithmetic series as well.

1.7 Conclusion

Despite the fact that no coherent prime numbers pattern is known beyond 24, it is still possible to derive a general formula for calculating the probability of a prime number "occurring'' between any two known consecutive prime squares. Bernard Riemann (see proceedings of the Royal Society: The first digit frequencies of prime numbers and Riemann zeta zeros, by B. Luque and Lucas Lacasa) and Gauss each contributed significantly towards calculating the distribution of prime numbers. We will use the pattern of the row of remainders at the end of this book to derive both men's formulae.

As an example, we calculate the probability that 13 is prime(13 lies between the prime squares 9 and 25) as follows:

The probability that a remainder in the first column of remainders between 9 and 25 is not zero is 1/2 since in the first column (under bold 2 of table 1.2) there are only two possibilities; 0 or 1. In the second column there are three possibilities; 0, 1 and 2, therefore the probability that a remainder in the second column is not 0 is $2/3$. For an integer to be prime both remainders in both columns must not be zero. Therefore the probability that both remainders of 13's row of remainders are not zero in both columns is $\frac{1}{2} \times \frac{2}{3} = \frac{2}{6}$. So the probability that 13 or any integer from 9 to 24 is prime is $\frac{2}{6}$. The theoretical probability of finding a prime integer between 24 and 49 is $\frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} = \frac{8}{30}$

For interest sake, the theoretical probability that a number from 4 to 8 is prime is $\frac{1}{2}$ whereas the experimental one is $\frac{2}{5}$. The table 1.5 shows comparisons between experimental, theoretical probabilities, and $\frac{1}{lnx}$ between indicated **consecutive prime squares**. (Experimental probability is obtained by dividing the difference between any two consecutive prime squares by the number of prime numbers within that range)

The general formula for the theoretical probability is as shown below:

$$\frac{a-1}{a} \times \frac{b-1}{b} \times \frac{c-1}{c} \ldots \frac{\sqrt{x}-1}{\sqrt{x}} = \left(1-\frac{1}{a}\right) \times \left(1-\frac{1}{b}\right) \times \left(1-\frac{1}{c}\right) \ldots \times \left(1-\frac{1}{\sqrt{x}}\right) \sim \frac{1}{lnx}.$$

NB. The above equation is a form of Leonard Euler's product formula, where a=2, b=3, c=5 and so on (2, 3, 5 and $\sqrt{x}$ are the bold numbers at the top of all the columns of remainders between the two consecutive prime squares of interest, see pattern at the end of this paper). As an example, the theoretical probability of finding a prime number between 49 and 120 is $\frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} = \frac{24}{105}$.

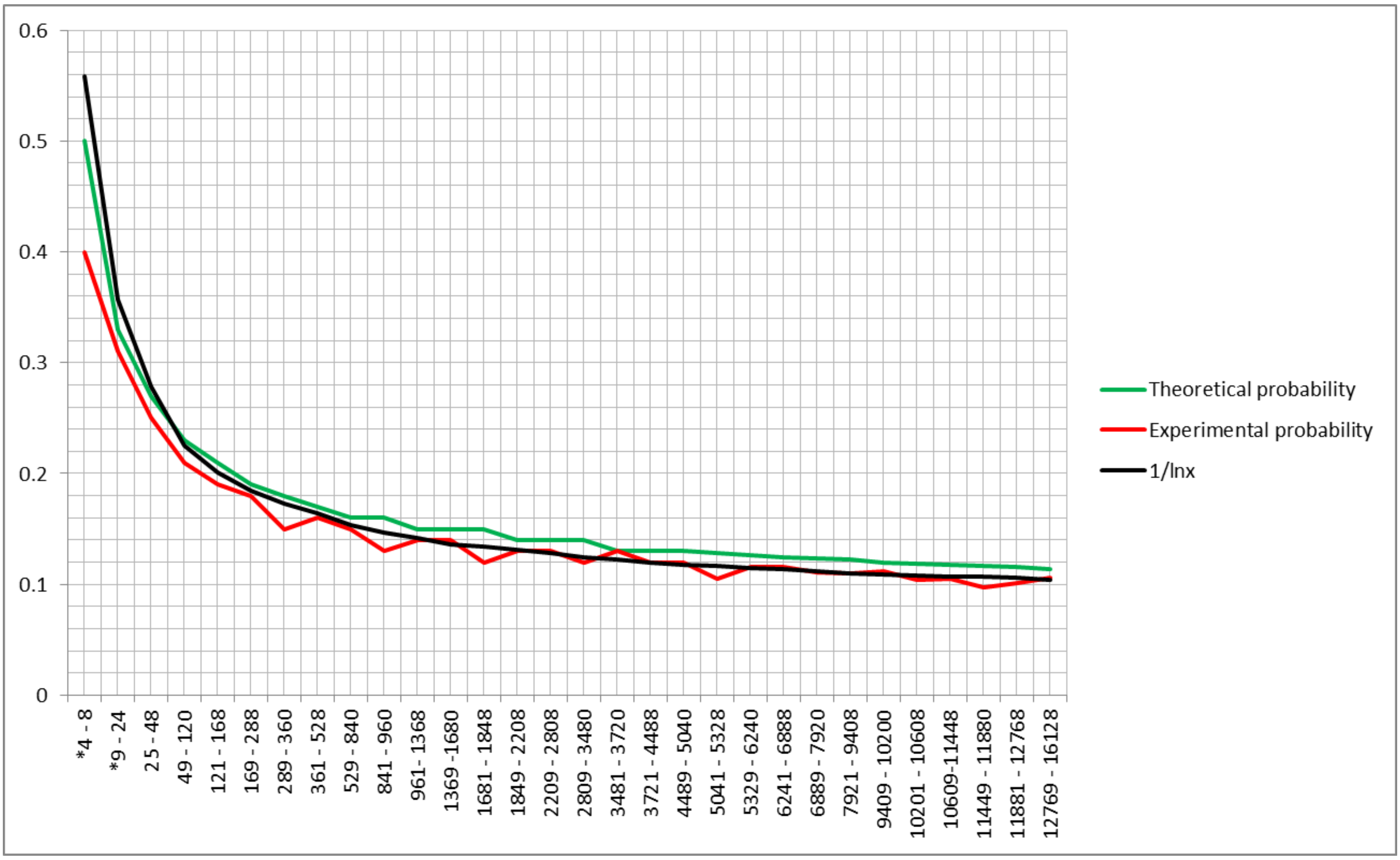
It is obvious that as we continue to multiply more and more fractions less than one, the product approaches zero. That is what Gauss observed in his analysis of the prime numbers.

range	Theoretical probability	Experimental probability	$\frac{1}{lnx}$
4 - 8	0.5	0.4	0.558
9 - 24	0.33	0.31	0.357
25 - 48	0.27	0.25	0.278
49 - 120	0.23	0.21	0.225
121 - 168	0.21	0.19	0.201
169 - 288	0.19	0.18	0.184
289 - 360	0.18	0.15	0.173
361 - 528	0.17	0.16	0.164
529 - 840	0.16	0.15	0.153
841 - 960	0.16	0.13	0.147
961- 1368	0.15	0.14	0.142
1369 -1680	0.15	0.14	0.136
1681 - 1848	0.15	0.12	0.134
1849 - 2208	0.14	0.13	0.131
2209 - 2808	0.14	0.13	0.128
2809 - 3480	0.14	0.12	0.124
3481 - 3720	0.13	0.13	0.122
3721 - 4488	0.13	0.12	0.120
4489 - 5040	0.13	0.12	0.118
5041 - 5328	0.128	0.105	0.117

5329 - 6240	0.126	0.116	0.115
6241 - 6888	0.124	0.116	0.114
6889 - 7920	0.123	0.111	0.112
7921 - 9408	0.122	0.11	0.110
9409 - 10200	0.120	0.112	0.109
10201 - 10608	0.119	0.104	0.108
10609-11448	0.118	0.105	0.107
11449 - 11880	0.117	0.097	0.107
11881 - 12768	0.116	0.101	0.106
12769 - 16128	0.114	0.106	0.104

Table 1.5

Fig 1.3



The graph above is a comparison of the experimental probability, the theoretical one, and of Gauss's equation $\frac{1}{lnx}$. We have modified Gauss's equation so that instead of just dividing positive integer *n* by *lnx*, the chosen range is only between two consecutive prime squares just like we do with the **theoretical probability** formula. If you compare the values of table 1.5, $\frac{1}{lnx}$ is an excellent approximation or best fit of the (oscillating) experimental probability graph. To calculate $\frac{1}{lnx}$, for any range of prime squares, we take the average of those consecutive prime squares and calculate their natural logarithm. The reciprocal of lnx is the probability of finding a prime number within that region between any two integers.

Nevertheless the theoretical approximation is also a good approximation of the prime numbers distribution but our inclusion of it is so that the reader can see the origins of Gauss and Riemann's equations.

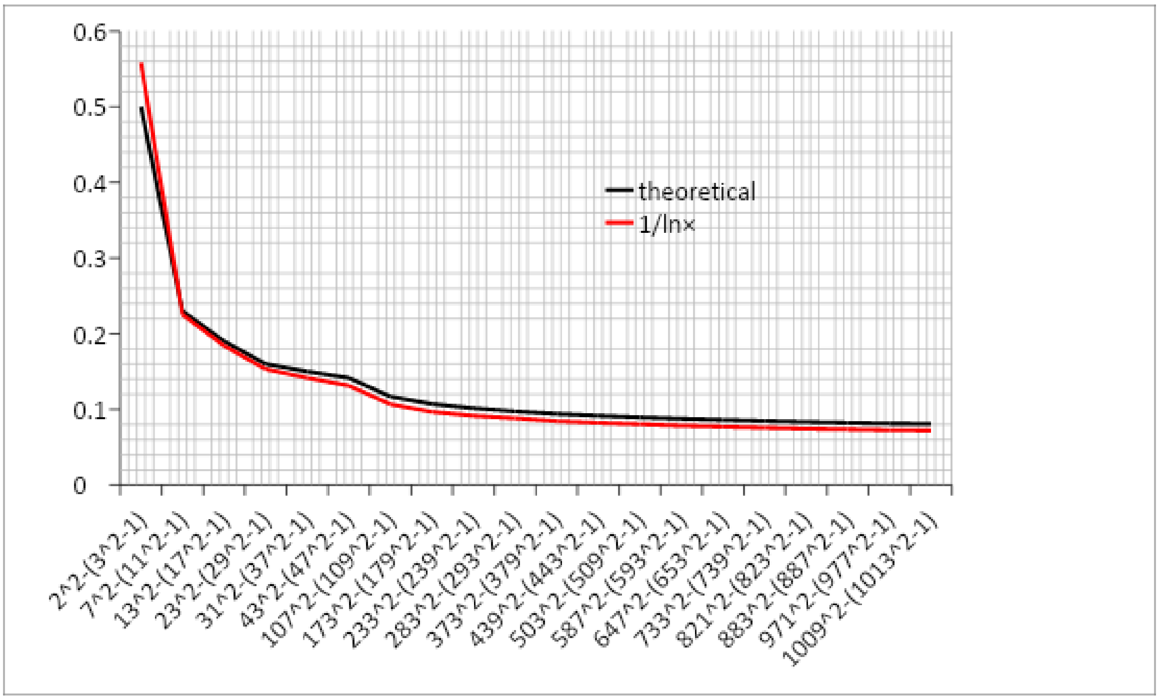


Fig.1.4

Figure 1.4 is a plot of probabilities between randomly selected consecutive prime pairs using equation $\frac{1}{lnx}$ and the theoretical one. As you can see they are fairly parallel nevertheless the former is always a better approximation.

range	**theoretical**	$\frac{1}{lnx}$
2^2-(3^2-1)	0.5	0.56
7^2-(11^2-1)	0.23	0.23
13^2-(17^2-1)	0.19	0.18
23^2-(29^2-1)	0.16	0.15
31^2-(37^2-1)	0.15	0.14
43^2-(47^2-1)	0.1417	0.1313
107^2-(109^2-1)	0.1169	0.1068
173^2-(179^2-1)	0.10720	0.0967
233^2-(239^2-1)	0.10159	0.09151
283^2-(293^2-1)	0.09745	0.08829
373^2-(379^2-1)	0.09405	0.08432
439^2-(443^2-1)	0.09154	0.08211
503^2-(509^2-1)	0.08943	0.08030
587^2-(593^2-1)	0.08766	0.07837
647^2-(653^2-1)	0.08611	0.07720
733^2-(739^2-1)	0.08462	0.07574
821^2-(823^2-1)	0.08332	0.07450
883^2-(887^2-1)	0.08225	0.07369
971^2-(977^2-1)	0.08129	0.07266
1009^2-(1013^2-1)	0.08088	0.07226

Table 1.6

If you go back to table 1.5 and compare $\frac{1}{lnx}$ versus the experimental probability you cannot help to be impressed by its accuracy. Multiplying the difference between two consecutive prime squares by the probability corresponding to the range between those prime squares gives you an approximation of the number of prime numbers. For example the logarithmic probability between 16128 and 12769 is 0.104 whereas the theoretical one is 0.114. Since the difference between 16129 and 12769 is 3360. Multiplying 3360 by each probability gives 383 and 349 respectively. The actual number of prime numbers between this range is 357! The reason why the theoretical method is important is that it leads to Bernard Riemann's formula as we will prove below.

Going back to the Gauss's equation, we see that the number of prime numbers between any two prime squares N^2 and n^2 is;

$\frac{1}{lnN^2}(N^2 - n^2),\ where\ \frac{1}{lnN^2}$ is the probability of finding a prime number within two consecutive prime squares (which is also the method you use to estimate the number of prime squares using the theoretical probability method).

$\frac{1}{lnN^2}(N^2 - n^2)$ can be re-written as $\frac{1}{lnN^2}(N + n)(N - n)$

As $N\ and\ n$ get closer and closer $N + n$ approaches $2N$ while $N - n$ reduces to dN

Thus equation $\frac{1}{lnN^2}(N^2 - n^2)$ becomes

$$\int_{n^2}^{N^2} \frac{1}{lnN^2} 2NdN.$$

Taking $N^2 = t, and\ dt = 2NdN\ (\ where\ lnN^2 \geq 2, since\ 2\ is\ the\ smallest\ prime)$we see that

$$\int_{n^2}^{N^2} \frac{1}{lnN^2} 2NdN = \frac{dt}{lnt},$$

which is the logarithmic function deduced by Riemann. Nevertheless $li(N^2) - li(n^2)$ is always almost equal to $\frac{1}{lnN^2}(N^2 - n^2)$ as can be seen in the example below.

We want to use equation $\frac{1}{lnN^2}(N^2 - n^2)$ and li(x) to calculate the number of prime numbers between consecutive two prime squares 16127^2 (260 080 129) and 16111^2 (259564321). In the former equation we find $\frac{1}{lnx}$ where *x* is the average of the two prime squares. $\frac{1}{lnx} = 0.0516115$

$(16127^2 - 16111^2) \times 0.0516115 = 26\ 621$ prime numbers

Li(260 080 129) - li(259 564 321)=26 621 prime numbers

Despite equal results above, Riemann's equation is the best in that it gives a good approximation of prime numbers between **any** two integers you can think of even if they are not consecutive prime squares. The author concludes this paper hoping that the reader has understood and appreciates the ultimate prime numbers algorithm that explains why prime numbers behave the way they do. Prime numbers are not a mystery but merely special members of arithmetic series patterns. The remainders pattern might be tedious but it gives the reader an insight about the nature of prime numbers and hopefully it would enable some researchers to solve their own prime numbers problems.

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