Construction with ruler and compass, van Hiele model for geometry thinking and Project-based learning

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Abstract

We describe how problems of geometric construction using straightedge and compass can be introduced to students through project-based learning. We discuss how these problems can be extended to the upper half-plane model. Furthermore, we discuss the use of these problems to assess advanced levels in van Hiele model for geometry thinking.

Keywords: Geometry; construction; van Hiele model; project-based learning;

Introduction

Geometry is a fundamental branch of mathematics with tight connections to real life. It is mainly concerned with the study of the shape of individual objects, spatial relationships among various objects, and the properties of surrounding space [6]. Geometry is taught almost at all levels of curricula, from primary schools where basic shapes and elementary rules of computing areas are introduced to graduate level where different models and branches of modern geometry are studied. Teaching axiomatic geometry is generally a challenging and demanding task. This is mainly due to the nature of the subject. Indeed, unlike other branches of mathematics, geometry is not algorithmic and each particular situation requires a particular type of reasoning. According to Artmann [1], "Rather than the memorization of simple algorithms to solve equations by rote, it demands true insight into the subject, clever ideas for applying theorems in special situations, an ability to generalize from known facts." It is worth mentioning that geometry teaching and learning remains a controversial issue as there is no agreement among educators about the most effective teaching approach. A considerable amount of research work in mathematics education has been done to address issues related to the teaching and learning of geometry [4].
Geometric constructions using straightedge (non graduated ruler) and compass represent a fundamental topic in geometry learning. These construction problems are of central importance in Euclidean geometry. According to Berinde [2], problems of geometric constructions using ruler and compass, or only ruler, form a very special class of problems which, in order to be solved, require not only a very good knowledge of basic results in geometry but also special skills and cleverness. These construction problems range from trivial ones, that can be performed in absolute geometry, with very elementary justifications to more advanced ones which require higher logical reasoning skills. These problems can be extended to other

types of geometries as well. In particular, to hyperbolic geometry in any of its models. Overall, they are suitable tools to assess students thinking at level 3 and 4 in the van Hiele model as we shall explain later.

In this paper, we describe how these construction techniques and skills can be introduced to students through project-based learning. More precisely, we give explicit examples of such constructions that can furnish a project suitable for high school students or fresh undergraduate students. We also discuss how these constructions fit in the well-known van Hiele model for geometry thinking.

The van Hiele Model

The van Hiele model for geometric thinking was introduced by the Dutch scholar Pierre van Hiele and his wife Dina van Hiele-Geldof in the middle of the twentieth century. The development of this model emerged through the challenges faced by the couple in teaching geometry to their students. It was first applied in the revision of the geometry curricula in the former Soviet Union. Then, the model slowly gained attention and was considered in other educational systems. Experimental research provided evidence that supports accuracy of the model in assessing students geometry learning [3]. The model is recognized by mathematics educators as the most well-defined theory addressing students' development of geometry thinking [5,8].

The main feature of the van Hiele theory is that it divides geometry thinking stages attained by geometry learners into five different levels. Furthermore, it gives a description of how students progress from one level to the other. The theory identifies also a list of properties of these levels. We shall now list the five thinking stages that constitute the model.

- Level 0: (Basic Level), Visualization. At this level, students use visualization to recognize geometric figures by their shapes. Only nonverbal thinking is used at this stage.
- Level 1: Analysis. At this level, students recognize shapes by their properties. They start analysing and naming properties of geometric figures without seeing the relationships between them without giving any justification.
- Level 2: Abstraction, Informal Deduction. At this stage, students start to think abstractly and they are able to give simple arguments to justify their reasoning.
- Level 3: Deduction. At this level, students are able to give deductive proofs. They can use logical arguments to justify geometric results.
- Level 4: Rigor. At this final level, students are able of writing formal rigorous proofs. In addition, they understand axiomatic systems and differentiate between Euclidean and non-Euclidean geometries.

Construction Projects

In this section, we shall describe the type of projects that can be assigned to high school students at their terminal year or fresh undergraduate students to learn the fundamental geometric skill of construction using ruler and compass. Pre-service teachers are definitely supposed to master these techniques and to understand the supporting algebraic theory that stands behind these problems. We will divide these constructions into 3 types. The main component in such project is made up of classical constructions which require only basic Euclidean geometry knowledge. Furthermore, we suggest the augmentation of these projects by creative constructions and some examples of constructions in the upper half-plane model for hyperbolic geometry ***3.1 Classical constructions***

We split these classical constructions into four sets depending on their complexity and the type of arguments used to justify the construction.

Set 1. These basic constructions can be performed using only the first four axioms of Euclid. So they are done in absolute (neutral) geometry. The first problem is straightforward. The second one is multi-step, but depends only on the first problem. The third is a consequence of the two previous ones.

1. Construct an equilateral triangle given one side; (Euclid's Proposition 1)
2. Reproduce (copy) a given line segment at a given point; (Euclid's Proposition 2)
3. Given two inequal line segments. Cut off from the greater a line segment equal to the smaller; (Euclid's Proposition 3)

Set 2. The justification of these constructions requires the use of congruent triangles.

1. The bisector of an angle; (Euclid's Proposition 9)
2. The perpendicular bisector of a line segment; (Euclid's Proposition 10)
3. The perpendicular to a line through a given external point; (Euclid's Proposition 11)
4. The parallel to a line through a given external point; (Euclid's Proposition 10)

Set 3. These constructions use Euclid's fifth axiom, the parallel postulate, and the notion of similar triangles.

1. Tangent line to a circle through an external point;
2. The symmetric to a point with respect to a given circle, (the image of a point by an inversion).
3. Divide a given line segment into n equal parts;

Set 4. Here are examples of other constructions which require more careful proofs.

1. Tangent line to two given circles;
2. A square inscribed in a given circle;
3. A regular hexagon inscribed in a given circle;
4. A regular hexagon given one side;

Example 1. We shall now give the solution of problem 4 of Set 2. Our objective is to understand which level of van Hiele model can be assessed using such questions.

Given a line m and an external point A. Let us construct the parallel to m through A. The steps of this construction are summarized below, see Figure 1.

- Construct a line n through A which meets the line m at a point B.
- Construct an arc of circle centered at B which intersects the line n at Q and line m at P.
- Construct an arc centered at A having radius BQ. It intersects line n at a point F.

- Construct an arc centered at F and having radius QP and an arc centered at A and having radius BP. The two arcs meet at a point G.
- Draw the line (AG).

Claim. Line (AG) is the parallel to m through A.

Justification. By construction, we have BQ=AF, QP=FG and BP=AG. Consequently, the two triangles BQP and AFG are congruent by the side-side-side congruence property. We conclude then that the angles PBQ and GAF are equal. Hence, the lines (BP) and (AG) are parallel.

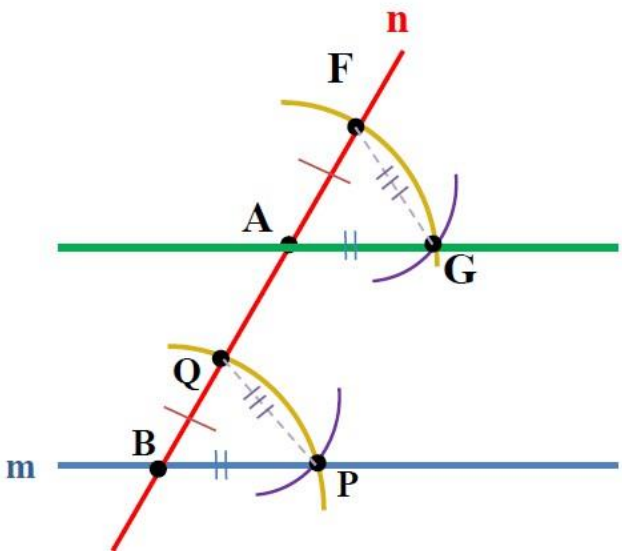

Figure 1. Construction of the parallel to a given line through an external point.

Remark. The construction above can be used to assess students' geometric ability to make formal deduction and to write complete proofs. This corresponds mainly to level 3 in van Hiele theory. In our opinion, all the other constructions above correspond to the same level. To better assess level 4 of the model, we need to include more advanced construction problems.

Creative Constructions

Students' innovative and critical thinking can be assessed through implementing a creative component in the project. For instance, students can be asked to create new designs and construct them using ruler and compass. The creative designs can be of artistic, architectural, industrial or any other type. It should be nontrivial and executed only using the ruler and compass. From personal experience, although students are able of creating original designs, they often find it difficult to mathematically justify some features of their own construction.

Constructions in Hyperbolic Geometry

The construction skills gained through the Euclidean examples discussed above can be consolidated by more advanced constructions in hyperbolic geometry. For instance, one may consider some constructions in the Poincaré upper half-plane model. To introduce this model to students, one need only some basic integration techniques that are often covered in high school mathematics curriculum. Indeed, the suggested constructions do not require any deep knowledge of the model, but some understanding of the different types geodesics [7]. Similar constructions can be performed also in the unit disk model or in other geometry models. Below, we give few simple examples of constructions that can be performed in the upper half-plane model.

1. Constructing straight lines; (geodesics)
2. Given two points I and P. Construct the hyperbolic circle centered at I and going through P;
3. Find the hyperbolic center of a given circle in the upper half-plane.

Example 2. We shall here give the solution of problem 2 above to show that it requires only basic knowledge of the upper half-plane model for hyperbolic geometry. Indeed, there are two cases to be considered depending on whether line (IP) is vertical or not. If (IP) is not a vertical line, the steps of the constructions are summarized below, see Figure 2.

- Construct the bowed geodesic IP and let C be its center. This point is actually the intersection of the Euclidean perpendicular bisector of line segment IP and the x-axis.
- Construct the tangent line m to the bowed geodesic IP at P. It intersects the x-axis at a point S.
- Draw the vertical line through I. It will meet the line m at a point Q and the x-axis at a point R.

Claim. The Euclidean circle centered at Q and through P is the hyperbolic circle centered at I and through P.

Justification. The proof of the claim above requires only the use of the properties of geodesics in hyperbolic geometry and the fact that a hyperbolic circle is indeed an Euclidean circle with different center and different radius.

Recall that an Euclidean circle of center (h,k) and radius r, with $r < k$ is the same as the hyperbolic circle of center (h,K) and radius R where

$$K = \sqrt{k^2 - r^2} \text{ and } R = \frac{1}{2}\ln\left(\frac{k+r}{k-r}\right).$$

Hence, in our case, it will be enough to prove that $RI^2 = RQ^2 - QP^2$.

Notice that by the Pythagorean theorem applied to right-angled triangles QCP and QCR we have: $QC^2 = CP^2 + QP^2 = QR^2 + RC^2$.

Since $CP = CI$, we get $RC^2 + RI^2 + QP^2 = QR^2 + RC^2$, which implies that $RI^2 + QP^2 = QR^2$.

Finally $RI = \sqrt{QR^2 - QP^2}$. This completes the proof.

In the case (IP) is a vertical line. There are two subcases to be considered depending on the position of I and P. The two subcases can be done similarly. We explain the construction steps assuming that I is located above P.

- Let C be the intersection of the line (IP) and the x-axis. Construct the Euclidean circle q centered at C and of radius CI.
- Draw the horizontal line through P. It interests the circle q at a point Q.
- Let m be the tangent line to the circle through Q.
- Extend the line m to meet the line (IP) at a point S.

- Construct the Euclidean circle centered at the midpoint of line segment PS and point P. This is the hyperbolic circle of center I and through P.

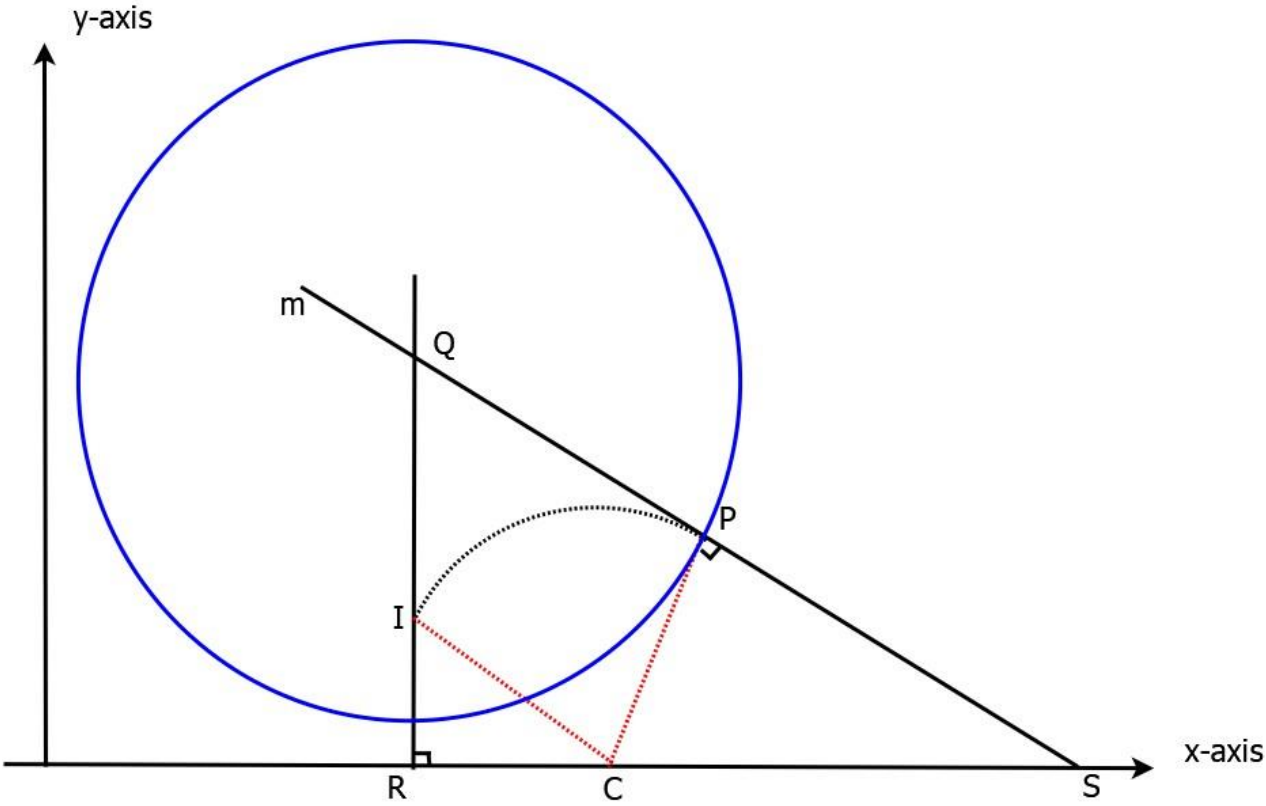

Figure 2. Construction of a hyperbolic circle given its hyperbolic center and a point.

Conclusion

In this paper, we described how problems of construction using ruler and compass can be introduced to students through project-based learning. These projects can be augmented by creative and hyperbolic geometry constructions to assess higher level of van Hiele model for geometry thinking.

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