# Solving Cubic Equations Using Direct Factoring in Complex Field

## Abstract

*Cardano's Method is not easily understood by undergraduate students. In this research project, we developed a method that students can understand without advanced mathematics skills. The method we developed only need to use the skills of factoring polynomials in complex field and finding cubic roots of a complex number.*

The procedures we developed are as following:

1) Write cubic equation in the form of $A^3 + B^3 + C^3 – 3ABC = 0$, where A is a function of x, B and C are complex numbers.
2) Solve the quadratic equation $Z^2 – (B^3 + C^3)Z + B^3 C^3 = 0$, which gives B and C.
3) Factor equation of (1) into $(A + B + C)(A +Bw + Cw^2)(A +Bw^2+Cw) = 0$, where w is a complex root of 1.
4) Solve equations $A+B+C = 0$, $A+B\omega+C\omega2 = 0$, and $A+B\omega2+C\omega = 0$

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The Cardano's Method

Let the cubic equation $ax^3+bx^2+cx +d = 0$. We want to find all roots of the equation.

1. Reduce the equation to $x^3+px + q =0$
2. Substitute x by y + z. We obtain the equation $y^3+z^3+(3yz + p)(y +z) + q = 0$
3. Set $3yz + p = 0$ the equation at step (2) becomes $y^3+z^3+ q = 0$.
4. Solve the system of equations at step (3). We get six pairs of solution for y. Let $y = \frac{-p}{3z}$ and substitute to the second equation. This gives $27z^6+27qz^3-p^3=0$
5. Solve the 6th degree equation from step 4 and had six solutions for z.
6. Choose appropriate y and z from step 4 and step 5 to get the solution for x.

Those procedures are not easily understood by college students especially at step (3). Many students will ask why can we set $3yz + p = 0$. Also at step (6), the 6th degree equation usually kicks students out of the classroom. The method that I am going to present here is much easier for student to understand. The foundation of this method is factoring a polynomial in complex number field and finding cubic roots of a complex number.

**The Direct Factoring Method**

First of all, let us recall some facts from Precalculus about polar form of complex numbers.

Definition The polar form of the complex number $z = a + bi$ is given by $z = r(\cos \theta + i\sin \theta)$ where $a = r\cos \theta$ and $b = r\sin \theta$, $r^2= a^2+ b^2$, and $\tan \theta = b/a$. We also use the notation $e^{i\theta}$ for the polar form, i.e. $= r(\cos \theta + i\sin \theta)= r e^{i\theta}$.

Theorem Nth Root of a Complex Number z

For a positive integer n, the complex $z = r(\cos \theta + i\sin \theta)$ has exactly n distinct roots given by $r^{1/n}$ $[\cos(\theta+2k\pi)/n + i\sin (\theta+2k\pi)/n]$ where $k = 0, 1, 2, \ldots, n-1$. In particularly the cubic root of 1 is 1, $\cos (\pi/3) + i\sin (\pi/3)$, and $\cos (2\pi/3) + i\sin (2\pi/3)$.

Lemma 1 Every cubic equation can be written in the form of $A^3+B^3+C^3-3ABC = 0$

Proof Let the cubic equation be $f(x) = x^3+px^2+qx+r = 0$

$$F(x) = (x + \frac{p}{3})^3+\frac{9q-3p^2}{9}(x + \frac{p}{3}) + \frac{27r-9pq+2p^3}{27} =0$$

Let $A = x + \frac{p}{3}$, $B^3+C^3=\frac{27r-9pq+2p^3}{27}$, $BC = \frac{3q-p^2}{9}$

$F(x) = A^3+ B^3+C^3-3ABC = 0$

Remark. We know that If α, and β are roots of the quadratic equation $x^2- bx + c = 0$, then $\alpha+ \beta = b$ and $\alpha\beta = c$. Therefore $B^3$ and $C^3$ are the roots of the quadratic equation $z^2-( B^3+C^3)z + B^3C^3= 0$ subject to $BC = \frac{3q-p^2}{9}$

Lemma 2 $A^3+B^3+C^3-3ABC = (A+B+C)(A^2 + B^2 + C^2 -AB -BC - AC)$

$= (A+B+C)(A+B\omega+C\omega^2)(A+B\omega^2+C\omega)$ where $\omega = \cos\frac{2\pi}{3} +i\sin\frac{2\pi}{3}$

**Procedure of Direct Factoring**

Step 1: Rewrite the equation in the form of $A^3+B^3+C^3-3ABC = 0$
Step 2: Solve the quadratic equation $z^2-( B^3+C^3)z + B^3C^3= 0$
Step 3: Factor the equation into $(A+B+C)(A+B\omega+C\omega^2)(A+B\omega^2+C\omega)= 0$
Step 4: Solve equations $A+B+C = 0$, $A+B\omega+C\omega2 = 0$, and $A+B\omega2+C\omega = 0$

Let me use an example to demonstrate the difference between these two methods.
Example: Solve the cubic equation $x^3-6x^2+11x -6 =0$

**Cardano's Method**

1. By shifting the roots to the right 2 units, we have the new equation $x^3-x = 0$
2. Replace x by y + z, we get $y^3+z^3 +(3yz - 1)(y + z) = 0$
3. Let $3yz - 1 = 0$ or $y = \frac{1}{3z}$ and plug in the equation at (2). We have $27z^6 + 1 = 0$
4. Solve the equation $27z^6 + 1 = 0$, we get $z^3= \frac{\sqrt{3}i}{9} = \frac{\sqrt{3}}{9}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$ or $z^3= -\frac{\sqrt{3}i}{9} = \frac{\sqrt{3}}{9}\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$
5. Finally $z_0=\sqrt[3]{\frac{\sqrt{3}}{9}}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$, $z_1=z_0\omega$, $z_2=z_0\omega^2$ and $z_4 = \sqrt[3]{\frac{\sqrt{3}}{9}}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$, $z_5 = z_4\ \omega$, $z_6 = z_4\ \omega$; where $\omega = \cos(\pi/3) + i\sin(\pi/3)$.
6. Plug solutions from (5) to $y = \frac{1}{3z}$, we get solutions of y.
7. X = ys + zs are solutions of the equation.

**Direct Factoring Method**

Step 1. We use synthetic division three times to convert the equation in the form of x – 2.

$x^3$-6$x^2$+11x -6 = $(x – 2)^3$+0$(x – 2)^2$- (x – 2) + 0 =0

A = x – 2, $B^3$+ $C^3$ = 0, BC = $\frac{1}{3}$

Step 2. We solve the z-equation $z^2$-0z + $\frac{1}{27}$ = 0, Z = $\frac{\pm 1}{\sqrt{27}}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

Let $B^3 = \frac{1}{\sqrt{27}}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$ and $C^3 = \frac{-1}{\sqrt{27}}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

We have three roots for B and C respectively and denoted by

$B_0 = \frac{1}{\sqrt{3}}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$, $B_0\omega$, $B_0\omega^2$

$C_0 = -\frac{1}{\sqrt{3}}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$, $C_0\omega$, $C_0\omega^2$.

By checking the condition BC = $\frac{1}{3}$, we know that B = $B_0 = \frac{1}{\sqrt{3}}e^{i\frac{\pi}{6}}$ and C = $C_0\omega = \frac{-1}{\sqrt{3}}e^{i\frac{5\pi}{6}}$

Step 3. 0 = $x^3$-6$x^2$+11x -6 = (A+B+C)(A+B$\omega$+C$\omega^2$)(A+B$\omega^2$+C$\omega$)

$$=\left(x-2+\frac{1}{\sqrt{3}}(e^{i\pi/6}-e^{i5\pi/6})\right)\left(x-2+\frac{1}{\sqrt{3}}e^{i(\pi/6+2\pi/3)}+\frac{-1}{\sqrt{3}}e^{i(5\pi/6+4\pi/3)}\right)$$

$$\left(x-2+\frac{1}{\sqrt{3}}e^{i(\pi/6+4\pi/3)}-\frac{1}{\sqrt{3}}e^{i(5\pi/6+2\pi/3)}\right)$$

= (x – 2 + 1)(x – 2 – 1)(x – 2 + 0)

= (x – 1)(x – 2)(x – 3) = 0

Therefore x = 1, x = 2 or x = 3