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A note about a new method for solving Riccati differential equations

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Abstract

Al Bastami, Belić, and Petrović (2010) proposed a new method to find solutions to some Riccati differential equations. Initially, they obtain a second-order linear ordinary differential equation (ODE) through a standard variable change in the Riccati equation. They then propose a new variable change and discuss the resolution of the resulting ODE in two cases. In the first one, the resulting ODE has constant coefficients. In the second case, they claim that it is possible to arbitrarily choose one of the resulting ODE coefficients and solve particular Riccati ODEs. We show in this work that all Riccati equations that belong to the first case can also be solved by Chini's method. Furthermore, we show that any Riccati equation fits the second case and that the choice of the resulting ODE coefficients is not free.

Keywords: Riccati equation; Chini's method; new variable change.

1. Introduction

At low temperatures, the Gross-Pitaevskii equation (GPE) describes Bose-Einstein condensation of atomic Bose gases cooled and confined in traps. Al Bastami, Belić, and Petrović (2010) considered the generalized GPE in (3+1)-dimension for the Bose-Einstein condensate wave function $\psi(x, y, z, t)$, with distributed time-dependent coefficients

$$i\partial_t\psi + \frac{\beta(t)}{2}\Delta\psi + \chi(t)|\psi|^2\psi + \alpha(t)r^2\psi = i\gamma(t)\psi. \tag{1}$$

Here t is time; $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the three-dimensional Laplacian; $r = \sqrt{x^2 + y^2 + z^2}$ is the position coordinate, and $\alpha(t)$ stands for the strength of the quadratic potential as a function of time. The functions $\beta(t)$, $\chi(t)$, and $\gamma(t)$ stand for the diffraction, nonlinearity, and gain/loss coefficients, respectively.

Using Kamke's (1977) techniques to solve (1) requires solving several auxiliary problems. One of them requires determining the chirp function, which satisfies a Ricatti equation whose coefficients depend on $\alpha(t)$ and $\beta(t)$.

Al Bastami, Belić, and Petrović (2010) introduced a new method for solving a Riccati differential equation,

$$y' = P(x) + Q(x)y + R(x)y^2, \tag{2}$$

in closed form provided its coefficients $P(x)$, $Q(x)$, and $R(x)$ fulfill to required relations.

Using a standard change of variables $y = -u'/(uR)$ in (2), they reduced the Riccati equation to a second-order linear ordinary differential equation (ODE)

$$u'' + a(x)u' + b(x)u = 0, \tag{3}$$

where $a(x) = -(Q + R'/R)$ and $b = PR$.

Al Bastami, Belić, and Petrović (2010) assumed $b(x) = P(x)R(x) > 0$ and defined in (3) a new change of variables:

$$z \equiv z_o + s\int_{x_0}^{x}\sqrt{\frac{b(\varepsilon)}{B(\varepsilon)}}\,d\varepsilon, \tag{4}$$

where $s = \pm 1$ and $B(x)$ is an arbitrary function. Furthermore, taking the derivative of (4) and squaring the result, they deduced a relation between the new variable $z = f(x)$ and the arbitrary function $B(x)$:

$$B(x) = \frac{b(x)}{\left(\frac{dz}{dx}\right)^2}. \tag{5}$$

It follows from (3) and (4) that

$$\frac{d^2u}{dz^2} + 2A\frac{du}{dz} + Bu = 0, \tag{6}$$

where

$$2A = \frac{\frac{d^2z}{dx^2} + a(x)\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}.$$

Al Bastami, Belić, and Petrović (2010) pointed out the coefficients of ODEs (3) and (6) are closely related. To simplify their presentation, they denoted $c = \frac{b}{B}$. It follows from (4) that

$$\frac{dz}{dx} = sc^{1/2}, \qquad \left(\frac{dz}{dx}\right)^2 = \frac{b}{B} = c \qquad \text{and} \qquad 2\frac{dz}{dx}\frac{d^2z}{dx^2} = c'.$$

Therefore,

$$\left(\frac{b}{B}\right)' + 2a\left(\frac{b}{B}\right) - 4As\left(\frac{b}{B}\right)^{3/2} = 0. \tag{7}$$

Al Bastami, Belić, and Petrović (2010) considered two cases in (6): A and $B > 0$ are constants, and $A = 0$ and $B = B(x)$ is an arbitrary function.

Our main remarks about the method proposed by Al Bastami, Belić, and Petrović (2010) are:

1. For all Riccati equations that belong to the case A and $B > 0$ are constants in (6), it is also possible to solve them by Chini's method – see Chini (1924, 1925) or Kamke (1977) – (see Subsection 2.1).
2. For the second case, we claim that any Riccati equation belongs to the second case; furthermore, the function $B(x)$ cannot be arbitrarily chosen in (4) (see Subsection 2.2).

2. Comments on the new method

It is crucial to notice that $z = f(x)$ in (4) depends on the coefficients of the Ricatti equation (2) and that the choice of the function $B(x)$ completely defines the new variable $z = f(x)$. On the other hand, given $z = f(x)$, if $\frac{dz}{dx} \neq 0$, it is also possible to associate a function $B(x)$ to it through the relation (5). Such interdependence between $z = f(x)$ and $B(x)$ will be essential in analyzing the case $A = 0$.

2.1 Case A and $B > 0$ (constants) and Chini's method

When the coefficients of the ODE (6) are constant, it is easily solved. Al Bastami, Belić, and Petrović (2010) established a necessary condition for the coefficients A and $B > 0$ in (6) to be constants. They have used (7) and the connection between coefficients $a(x)$ and $b(x)$ in (3) and coefficients $P(x)$, $Q(x)$, and $R(x)$ of the Riccati equation to attain their goal.

Recall that $B > 0$ is a constant. Al Bastami, Belić, and Petrović (2010) multiplied (7) by B, to obtain

$$b' + 2ab = \frac{4sA}{\sqrt{B}} b^{3/2}.$$

That is,

$$\frac{b'(x) + 2a(x)b(x)}{[b(x)]^{3/2}} = \frac{4sA}{\sqrt{B}}. \tag{8}$$

Substituting $a(x) = -(Q + R'/R)$ and $b = PR$ in (8), they got

$$\frac{[P(x)R(x)]' - 2[Q(x) + (R'(x)/R(x))]P(x)R(x)}{[P(x)R(x)]^{3/2}} = \frac{4sA}{\sqrt{B}}. \tag{9}$$

Al Bastami, Belić, and Petrović (2010) noticed that if A and $B > 0$ were constants in (6), then:

- The ratio on the left side of (9) is constant.
- The ODE (6) can be easily solved, and the general solution of Riccati equation (2) is obtained in closed form.

Now, we present Chini's method. To solve an ODE in the form

$$y' = P(x) + Q(x)y + R(x)y^n, \tag{10}$$

Kamke (1977) cites two works from Chini (1924, 1925). If a constant α exists such that $w = (P(x)/R(x))^{1/n}$ is a solution of the linear equation

$$w' - Q(x)w = \alpha P(x), \tag{11}$$

then the change of variables $y = w(x)u$ transforms (10) into a separable ODE.

When $n = 2$, if there exists a constant α such that $w = (P(x)/R(x))^{1/2}$ is a solution of the linear equation (11), then the change of variables $y = w(x)u$ transforms the Riccati equation (2) into

$$P(x)(u^2 + 1) = w'u - Q(x)wu + wu',$$
$$P(x)(u^2 - \alpha u + 1) = wu',$$

which is a separable ODE.

In the case of Riccati equations, Cheb-Terrab, and Kolokolnikov (2008) claim that Chini's method is equivalent to verifying that

$$\mathcal{I} \equiv \frac{P'(x)R(x) - P(x)R'(x) - 2P(x)Q(x)R(x)}{[P(x)R(x)]^{3/2}} \tag{12}$$

is equal to a constant, where $P(x)$, $Q(x)$, and $R(x)$ are the coefficients of the Riccati equation. In fact, $\mathcal{I}$ equals to a constant in (12) if, and only if, $w = (P(x)/R(x))^{1/2}$ is a solution of (11) when $n = 2$ and $\alpha = \frac{\mathcal{I}}{2}$.

Notice that Chini's method is applicable for any Riccati equation (2) whose coefficients $P(x)$, $Q(x)$, and $R(x)$ are such that $\mathcal{I}$ in (12) is constant. That is, the change of variables $y = (P(x)/R(x))^{1/n}u$ transforms the Riccati ODE (2) into a separable one.

Now we analyze the connection between the case A and $B > 0$ (constants) in (6) and Chini's method in detail. Notice that (9) is equivalent to

$$\frac{P'(x)R(x) - P(x)R'(x) - 2P(x)Q(x)R(x)}{[P(x)R(x)]^{3/2}} = \frac{4sA}{\sqrt{B}}.$$

Therefore, anytime the new method proposed by Al Bastami, Belić, and Petrović (2010) is applied to Riccati equation (2) and results in constant coefficients A and $B > 0$ in (6), we always have constant $\mathcal{I}$ in (12). In this case, it is possible to solve (2) directly by Chini's method. That is, there is a change of variables that transforms (2) into a separable ODE. One of our claims is proved. For all Riccati equations that belong to the case A and $B > 0$ are constants in (6), it is also possible to solve them by Chini's method. In this case, Al Bastami, Bellic, and Petrovic's approach gives the solution in closed form.

2.1 Case $A = 0$ and $B = B(x)$

If $A = 0$, Al Bastami, Belić, and Petrović (2010) noticed that they might rewrite (6) as

$$\frac{d^2u}{dz^2} + B(z)u = 0,$$

where z is given by (4). They claim that $B(x)$ still is an arbitrary function, but they also require that

$$\frac{b}{B} = \left(\frac{b}{B}\right)_0 \exp\left(-2\int_{x_0}^{x} a\,dx\right), \tag{13}$$

where $a(x) = -(Q + R'/R)$ and $b = PR$ depends on the original coefficients $P(x)$, $Q(x)$, and $R(x)$ of the Riccati equation (2).

It follows from (13) that $B(x)$ is not an arbitrary function anymore. It is important to notice that in (4), $B(x) > 0$ is an arbitrary function. However, if $A = 0$ in (6), then we have to choose $B(x)$ in (4) such that

$$B(x) = \left(\frac{B}{b}\right)_0 b(x) \exp\left(2\int_{x_0}^{x} a\,dx\right). \tag{14}$$

That is, for $A = 0$, it is necessary that in $z = f(x)$ defined in (4), we take $B(x)$ as in (14).

Finally, we show that any Riccati equation (2) belongs to the second case. That is, for each Riccati ODE (2), it is always possible to choose in (4) a function $B(x)$ such that $A = 0$ in (6). Given a Riccati equation (2), proceed with the change of variables $y = -u'/(uR)$ to reduce it to a second-order linear ODE

$$u'' + a(x)u' + b(x)u = 0, \tag{15}$$

where $a(x) = -(Q + R'/R)$ and $b = PR$.

Let

$$B(x) = b(x) \exp\left(2\int_{x_0}^{x} a\, dx\right)$$

in (4) to obtain:

$$z = z_0 + s\int_{x_0}^{x}\sqrt{\frac{b(\xi)}{B(\xi)}}d\xi = z_0 + s\int_{x_0}^{x}\sqrt{\frac{b(\xi)}{b(\xi)\, exp\,\left(2\int_{x_0}^{\xi} a\,(\mu)d\mu\right)}}d\xi.$$

That is

$$z \equiv z_0 + s\int_{x_0}^{x}\exp\left(-\int_{x_0}^{\xi} a\,(\mu)d\mu\right)d\xi, \tag{16}$$

were $s = \pm 1$.

Notice that

$$\frac{du}{dx} = \frac{du}{dz}\frac{dz}{dx}, \tag{17}$$

$$\frac{d^2u}{dx^2} = \frac{du}{dz}\frac{d^2z}{dx^2} + \frac{d^2u}{dz^2}\left(\frac{dz}{dx}\right)^2. \tag{18}$$

Then, using (17) and (18) in (15), we obtain:

$$\left(\frac{dz}{dx}\right)^2\frac{d^2u}{dz^2} + \left[\frac{d^2z}{dx^2} + a(x)\frac{dz}{dx}\right]\frac{du}{dz} + b(x)u = 0.$$

Rewrite (15) in this new variable, to get:

$$\frac{d^2u}{dz^2} + 2A\frac{du}{dz} + Bu = 0, \tag{19}$$

where

$$2A = \frac{\frac{d^2z}{dx^2} + a(x)\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} \qquad \text{and} \qquad B = \frac{b(x)}{\left(\frac{dz}{dx}\right)^2}$$

It follows from (16) that

$$z' = s\exp\left(-\int_{x_0}^{x} a\,(\mu)d\mu\right),$$

$$z'' = -sa(x)\exp\left(-\int_{x_0}^{x} a\,(\mu)d\mu\right).$$

That is

$$\frac{d^2z}{dx^2} + a(x)\frac{dz}{dx} = 0,$$

and we conclude $A = 0$. That is, for A to be equal to zero it is necessary and sufficiente to choose $B(x)$ in (4) satisfying (14).

Thus, for any Riccati ODE (2), it is possible to choose in (4) a function $B(x)$ such that $A = 0$ in (6). Besides, if we take B in (4) that does not satisfy (14), we obtain $A \neq 0$ in (6).

We have shown our second claim. Any Riccati equation belongs to the second case; furthermore, the function $B(x)$ cannot be arbitrarily chosen in (4) when $A = 0$.

3. Final remarks

We presented an analysis of the new method for solving Riccati differential equations proposed by Al Bastami, Belić, and Petrović (2010). Their method considers two cases. All Riccati ODEs that belong to its first case, where A and $B > 0$ are constants, can also be solved by Chini's method. Their method has the advantage of presenting the solution in closed form. In its second case, where $A = 0$, it is not possible to arbitrarily choose the function $B(x)$. In addition, for any Riccati equation (2), with $P(x)R(x) > 0$, it is possible to choose $B(x)$ leading to the second case of the method presented in (2010). That is, any Riccati equation belongs to the second case; furthermore, the function $B(x)$ cannot be arbitrarily chosen in (4).

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