On the Well-ordering Principle and the Principle of Finite Induction

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Abstract

In this note the equivalence among the Well-ordering Principle, the Principle of Finite Induction and certain natural conditions concerning the set of integers is discussed, thereby clarifying facts encountered in the literature.

Keywords: set of natural numbers, set of integers, Well-ordering Principle, Principle of Finite Induction.

Introduction

Peano Postulates for the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of natural numbers [5, p. 35; 7], which may be regarded under the viewpoint of universality [3; 4; 5, Chap. 2], subsume the Principle of Finite Induction:

If $U$ is a subset of $\mathbb{N}$ such that $0 \in U$ and $n + 1 \in U$ whenever $n \in U$, then $U = \mathbb{N}$.

The Principle of Finite Induction ensures the validity of the Well-ordering Principle, which reads:

Every non-empty subset of $\mathbb{N}$ admits a least element.

This is precisely the statement of Proposition 7, p. 41 of [5], in whose proof one assumes the existence of a non-empty subset $V$ of $\mathbb{N}$ which does not admit a least element and one shows, by induction on $n$, that ”$x \in V$” implies ”$x \geq n$”, from which one arrives at a contradiction.

On the other hand, the Principle of Finite Induction is a consequence of the Well-ordering Principle, as Theorem 4, p. 10 of [1] guarantees. As a matter of fact, in the proof of the just mentioned result, one takes $U$ as above and assumes that $U \neq \mathbb{N}$, that is, $N^* = \mathbb{N} \setminus U \neq \emptyset$. If $m$ is the least element of $N^*$, $m-1 \in U$, and hence $m = (m-1)+1 \in U$, which cannot occur.

In this note, motivated by results appearing in Chapter I of [1] and Chapter 1 of [6], equivalent conditions to the above-mentioned principles will be discussed. Historical comments concerning Mathematical Induction may be found, for example, in [2] and [6].

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As always, $\mathbb{Z}$ will denote the set of integers.

First we shall prove a result motivated by Exercise 5, p. 11 of [1].

Proposition 2.1. The following conditions are equivalent:
Proof. (a) ⇒ (b): Let $S$ be a non-empty subset of $\mathbb{Z}$ admitting an upper bound $s$. Since

$$X = \{s - t; t \in S\}$$

is a non-empty subset of $\mathbb{N}$, (a) guarantees the existence of an element $u$ of $S$ so that $s - u \leq s - t$ for all $t \in S$. Thus $t \leq u$ for all $t \in S$, proving (b).

(b) ⇒ (c): Let $T$ be a non-empty subset of $\mathbb{Z}$ admitting a lower bound. Then the non-empty subset $-T = \{-t; t \in T\}$ of $\mathbb{Z}$ possesses an upper bound. Hence, by (b), there is a $v \in T$ such that $-t \leq -v$ for all $t \in T$, that is, $v \leq t$ for all $t \in T$. Therefore (c) is established.

(c) ⇒ (a): It suffices to observe that $\mathbb{N}$ has a lower bound.

Before proceeding, let us introduce a few notations. Indeed, for each $z \in \mathbb{Z}$ let us write $\mathbb{Z}_z^+ = \{t \in \mathbb{Z}; t \geq z\}$. Obviously, the mapping

$$\varphi_z: t \in \mathbb{Z}_z^+ \mapsto \varphi_z(t) = t - z \in \mathbb{N}$$

is bijective. Let us also write $\mathbb{Z}_z^- = \{t \in \mathbb{Z}; t \leq z\}$. Clearly $\mathbb{Z}^-_z = -(\mathbb{Z}_z^+)$ and $z$ is the least (resp. greatest) element of $\mathbb{Z}_z^+$ (resp. $\mathbb{Z}_z^-$).

![Figure 1: The mapping $\varphi_z$.](image)

Remark 2.2. For all $r, s \in \mathbb{Z}$, with $r < s$, the infinite sets $\mathbb{Z}_r^+$ and $\mathbb{Z}_s^+$ are quite similar, in the sense that
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\[ \mathbb{Z}_r^+ = \mathbb{Z}_s^+ \cup \{r, \ldots, s-1\} \]

Figure 2: The sets \( \mathbb{Z}_r^+ \) and \( \mathbb{Z}_s^+ \), \( r < s \).

Evidently we would have a similar remark for the sets \( \mathbb{Z}_z^- \).

In the example below we shall furnish an infinite family of infinite subsets of \( \mathbb{N} \), each of which does not coincide with a set \( \mathbb{Z}_z^+ \).

**Example 2.3.** For each prime natural number \( p \), let us consider the subset

\[ X_p = \{p, p^2, \ldots, p^n, p^{n+1}, \ldots\} \]

of \( \mathbb{N} \). Since

\[ p^{n+1} - p^n = p^n(p - 1) \geq p^n \geq 2 \]

for every integer \( n \geq 1 \), \( X_p \) is an infinite set whose least element is \( p \) and which does not coincide with a set \( \mathbb{Z}_z^+ \), and the distances between two consecutive elements of \( X_p \) may be taken as big as we wish. Moreover, if \( p, q \) are arbitrary prime natural numbers, with \( p \neq q \), then \( X_p \cap X_q = \phi \).

The next result was motivated by Exercise 4, p. 10 of [1] and Theorem 1.3.1, p. 25 of [6].

**Proposition 2.4.** The following conditions are equivalent:

(a') Principle of Finite Induction;

(b') for each \( z \in \mathbb{Z} \), if \( R \subset \mathbb{Z}_z^+ \), \( z \in R \) and \( n + 1 \in R \) whenever \( n \in R \), then \( R = \mathbb{Z}_z^+ \);

(c') for each \( w \in \mathbb{Z} \), if \( S \subset \mathbb{Z}_w^- \), \( w \in S \) and \( n - 1 \in S \) whenever \( n \in S \), then \( S = \mathbb{Z}_w^- \).

**Proof.** (a') \( \Rightarrow \) (b'): Put \( L = \varphi_z(R) ; L \subset \mathbb{N} \) and \( 0 = \varphi_z(z) \in L \) (since \( z \in R \)). If \( m \in L \) is arbitrary, \( m = \varphi_z(n) \) for a (unique) element \( n \) of \( R \). By hypothesis, \( n + 1 \in R \) and

\[ \varphi_z(n + 1) = (n + 1) - z = (n - z) + 1 = \varphi_z(n) + 1 = m + 1, \]
showing that \( m + 1 \in L \). Thus, by \((a')\), \( L = \mathbb{N} \), which is equivalent to \( R = \mathbb{Z}_x^+ \). Hence \((b')\) holds.

\((b') \Rightarrow (c')\): First \(-S \subset -(\mathbb{Z}_w^+) = (\mathbb{Z}_-w^+) \) and \(-w \in -S\). Moreover, if \( n \in S \) is arbitrary and \( m = -n, m + 1 = -n + 1 = -(n - 1) \in -S \), because \( n - 1 \in S \) by hypothesis. Therefore, by \((b')\), \(-S = \mathbb{Z}_-w^+\), which is equivalent to \( S = \mathbb{Z}_w^+ \) and proves \((c')\).

\((c') \Rightarrow (a')\): Let \( T \subset \mathbb{N} \) be such that \( 0 \in T \) and \( n + 1 \in T \) whenever \( n \in T \). Then \( 0 \in (-T) \subset (-\mathbb{Z}_0^+) = \mathbb{Z}_0^- \) and \( n - 1 \in (-T) \) if \( n \in (-T) \), and \((c')\) yields \(-T = \mathbb{Z}_0^-\), which is equivalent to \( T = \mathbb{Z}_0^+ = \mathbb{N} \) and proves \((a')\).

This completes the proof.

What we have seen may be summarized in

**Corollary 2.5.** The conditions \((a), (b), (c), (a'), (b')\) and \((c')\) are equivalent.

**Conclusion**

In this note the equivalence among the Well-ordering Principle, the Principle of Finite Induction and certain natural conditions has been established.

**References**


