Title: Structure Theorems of Specker Groups (I)

Dr. Shinemin Lin*
Savannah State University
lins@savannahstate.edu

Abstract

Specker groups are subgroups of group of all finite values sequences of integers. Fuchs ([5]) developed many theorems about Specker groups. Here I want to use lattice-ordered group approach to develop the lateral completion of Specker groups and Specker spaces. The main goals of this research paper are to prove theorem 2.9 and corollary 2.10.

Introduction

I gather some terms and fundamental results about l-groups. For additional background, refer to [1], [4], and [6]. I follow tradition and use additive notation for the group operation even though most of groups may not be abelian. A partially ordered group (po-group) is a group (G, +) that has a partially order, ≤, defined on it such that a ≤ b implies x + a ≤ x + b and a + x ≤ b + x for all a, b, x of G. If the order is also a lattice order or a total order, then G is called a lattice-ordered group (l-group) or a totally-ordered group (o-group), respectively. The symbols $\wedge$, *and* $\vee$ are used to denote the greatest lower bound and the least upper bound.

A subgroup A of G is an l-subgroup if A is also a sub-lattice of G. An l-subgroup A is convex, of 0 ≤ g ≤ q ∈ A implies g ∈ A. Suppose H is an l-subgroup of an l-group G, H is large in G or G is an essential extension of H if for each convex l-subgroup L≠ 0 of G, L ∩ H ≠ 0. An l-group G is Archimedean if 0 ≤ ng < h for all n ∈ N implies g = 0. For an Archimedean l-group G, if G admits no proper essential extension then G is essentially closed. An essentially closed essential extension of l-group G is called essential closure of G and is denoted by G^e. An l-group G is complete (laterally complete), if every bounded (disjoint) subset M of G^+, ∨ M ∈ G. An l-group H is a completion (lateral completion) of G if H is complete (lateral complete), and G is l-isomorphically dense in H, and no proper l-subgroup of H containing an l-isomorphic copy of G is complete (lateral complete). The completion (laterally completion, divisible hull) of G will be denoted by G^ (G^L, G^d). An element h ≤ g is called a component of g if (g – h) ∧ h = 0; an element g > 0 of G is called singular if for any 0 < x ≤ g then x is a component of g. An l-subgroup H of G is saturated if for all h of H, the components of h belong to H.
For g ∈ G, $g^+ = g \vee 0$ ($g^- = (-g) \vee 0$) is called the positive (negative) part of g, and $|g| = g^+ + g^-$ is called the absolute value of g. The principal convex l-subgroup of G that is generated by g is denoted by G(g), which is the subset G(g) = {x ∈ G| 0 ≤ |g|≤ n|g| for some 0 < n ∈ N}. Finally the notation $R \otimes_Z G$ represents the tensor product of R and G.

Specker Groups

Let I be any index set. The set of all functions f : I → Z such that f assumes but a finite number of distinct values in Z is clearly an abelian group, namely, the group of all bounded functions of integers. Manifestly, every f ∈ G can be uniquely written in the form

(1) f = $n_1 h_{x_1} + \ldots + n_k h_{x_k}$ (k ≥0) where $n_1, \ldots, n_k$ are nonzero distinct integer, the subset $x_1, \ldots, x_k$ of I are pairwise disjoint, while the h_xs are characteristic functions of subset x of I: $h_x(i) = \begin{cases} 1 \; if \;\; i \in x \\ 0 \; if \;\; i \notin x \end{cases}$

A subgroup S of G is said to be a Specker group if f ∈ S implies $h_{xi} \in S$ for all I = 1, …, k, where f is in the form of (1).

Let I be any index set. The set of all functions f : I → R such that f assumes but a finite number of distinct values in R is clearly an abelian group, namely, the group of all bounded functions of real numbers. Manifestly, every f ∈ G can be uniquely written in the form

(2) f = $r_1 h_{x_1} + \ldots + r_k h_{x_k}$ (k ≥0) where $r_1, \ldots, r_k$ are nonzero distinct reals, the subset $x_1, \ldots, x_k$ of I are pairwise disjoint, while the h_xs are characteristic functions of subset x of I: $h_x(i) = \begin{cases} 1 \; if \;\; i \in x \\ 0 \; if \;\; i \notin x \end{cases}$

A subgroup S of G is said to be a Specker space if f ∈ S implies $h_{xi} \in S$ for all I = 1, …, k, where f is in the form of (2).

Definition 1.0. A relative complemented distributive lattice with the least element is called a generalized Boolean algebra.

Proposition 1.1 ([4]) An l-group is a Specker group if and only if it is generated as a group by singular elements. In this case the set of singular elements forms a generalized Boolean algebra.

The following theorems are well-known and the main results in the topic of Specker groups.

Theorem 1.2. (Fuchs, [5]) For a subgroup S of the group G = $\prod_I Z$, the following conditions are equivalent:

(a) S is a Specker group;
(b) f ∈ S implies $h_{xi} \in S$, where x is the support of f;
(c) S is pure in G and is a subring of G.

Theorem 1.3. ([4]) Let G be a subgroup of $\prod_I Z$ that is generated by characteristic functions of subsets of I. G is an l-group if and only if the meet of any characteristic functions in G is also in G. In this case, the characteristic functions are precisely the singular elements of G and so G is a Specker group.

Lateral Completion of Specker Groups

Definition 2.1. A generalized Boolean algebra (or Boolean algebra) B satisfies the countable chain condition, denoted by CCC, if every pairwise disjoint subset is countable.

Remarks: Let B be a Boolean algebra, then

(a) B satisfies the CCC if and only if the completion B^ satisfies the CCC.
(b) Let [B] be the Specker space generated by B, then [B]^ = [B^]^ and hence $[B]^e = [B\text{^}]^e$, where $[B]^e$ is the essential closure of [B].

Proof: [B] is dense in [B^] hence [B]^ ⊂ [B^]^.

On the other hand, If g ∈ [B^] then g = $r_1x_1 + \ldots + r_nx_n$, where $r_i \in R$ and $x_i \in$ B^ for i = 1, 2, …, n. For each $1 \le i \le n$, $x_i = \vee\{y_{ij} \mid y_{ij} \in B^+ \text{ and } y_{ij} \le x_i\}$ so $x_i \in$ [B]^ which implies $r_ix_i \in$ [B]^ for aall I and hence g ∈ [B]^.

Now $[B^\wedge] \subset [B]^\wedge$ implies $[B^\wedge]^\wedge \subset [B]^\wedge$.
Finally $[B]^e = ([B]^\wedge)^L = ([B^\wedge]^\wedge)^L = [B^\wedge]^e$.

(c) $[B]^\wedge$ is not a Specker space.
(d) A complete Specker space is a cardinal sum of reals.

Let B be a generalized Boolean algebra and $B^\wedge$ be the completion of B. Let E be the set $\{ f = \vee n_i x_i \mid i \in I \subset N, x_i \in B^{u\wedge}$, and pairwise disjoint, $n_i \in Z\}$. For f, g ∈ E, then $f = \vee \{ n_i x_i \mid i \in I \subset N\}$, $g = \vee\{\{ n_j x_j \mid j \in J \subset N\}$, define

$f + g = \vee_I \vee_J (n_i + m_j)(x_i \wedge y_j) + \vee_I n_i(x_i \backslash \vee_J y_j) + \vee_J m_j(y_j \backslash \vee_I x_i)$ and $f \geq g$ if $\vee_I x \geq \vee_J y_j$ and for i, j, if $x_i \wedge y_j \neq 0$ then $n_i \geq m_j$. Then (E +) is an l-group.

Lemma 2.2. E is complete.

Proof: Let $F = \{ f_\alpha \mid \alpha \in A \}$ be a bounded subset of E. We want to prove the l.u.b. of F exists.

Case 1: If F is bounded b nx for all $\alpha \in A$, where $n \in N$ and $x \in B^{u\wedge}$, then f_α has finite ranges. So $f_\alpha \in [B^{u\wedge}]$ for all $\alpha \in A$ and hence $\vee_A f_\alpha \in E$.

Case 2: If F is bounded by $g = \vee_I \{n_i x_i \mid i \in I, n_i \in N, x_i \in B^{u\wedge}\}$, then for all $\alpha \in A$, that $f_\alpha \leq g$ implies the characteristic function on support $f_\alpha \leq \vee_I x_i$.

For each $i \in I$, for all $\alpha \in A$, define $g_\alpha = f_\alpha \wedge n_i x_i$ then $g_\alpha \leq n_i x_i$. By case 1, $g_i = \vee_A g_\alpha \in E$. Also if $i \neq j$ then $g_i \wedge g_j = 0$, hence $\vee_I g_i \in E$ and $\vee_I g_i \geq f_\alpha$ for all $\alpha \in A$, then $k \wedge n_i x_i \geq f_\alpha \wedge n_i x_i$. That implies $k \wedge n_i x_i \geq \vee_A(f_\alpha \wedge n_i x_i) = g_i$. That is to say $k \geq g_i$ for all $i \in I$, so $k \geq \vee_I g_i$. Therefore $\vee_A f_\alpha = \vee_I g_i \in E$.

Lemma 2.3. Let $\{n_\alpha x_\alpha \mid \alpha \in A, n_\alpha \in N, x_\alpha \in B^{u\wedge}\}$ be disjoint subsets of E, then $\vee_A n_\alpha x_\alpha \in E$.

Proof: For each $i \in N$, let $y_i = \vee\{ x_\alpha \mid n_\alpha = i\}$ then $y_i \in B^{u\wedge}$. Let $g_i = i\, y_i$ then g_i belongs to $[B^{u\wedge}] \subset E$ and if $i \neq j$ then $g_i \wedge g_j = 0$. Therefore $g = \vee g_i \in E$. If $h \in E$ and $h \geq n_\alpha x_\alpha$ for all $\alpha \in A$, first we can write $h = m_1 s_1 \vee m_2 s_2 \vee \ldots \vee m_t s_t \ldots$ If $s_t \wedge x_\alpha \neq 0$ then $m_t \geq n_\alpha$, so for $i \in N$ since $y_i \leq$ characteristic function on the support of h $(=\vee s_t)$, $y_i \wedge s_t \neq 0$ for some t. That implies $m_t \geq i$ and $h \geq g_i$, and hence $h \geq g$. This shows $g = \vee_A(n_\alpha x_\alpha) \in E$.

Lemma 2.4. E is laterally complete.

Proof: Let $\{0 < g_\alpha \mid \alpha \in A\}$ be disjoint subsets of E. For all $\alpha \in A$, let $g_\alpha = \vee_{I_\alpha} \{n_{\alpha i} x_{\alpha i} \mid n_{\alpha i} \in N, x_{\alpha i} \in B^{u\wedge}\}$. Then $M = \{n_{\alpha i} x_{\alpha i} \mid i \in I_\alpha, \alpha \in A\}$ is a disjoint subset of E. By lemma 2.3, $\vee g_\alpha \in E$.

Lemma 2.5. E is the lateral completion of [B]

Proof: Since [B] is dense in E, $[B]^L \subseteq E^L = E$. On the other hand, if $g \in [B]$, $g = \vee_I(n_i x_i)$ where $x_i \in B^{u\wedge} \subseteq [B]$ implies $g \in [B\}^L$ and hence $E \subseteq [B]^L$.

Corollary 2.6. If G is a Specker group then G^L is complete.

Remark: Let G = [B] be the Specker group generated by B. Then $[B^{u\wedge}]$ is the maximal essential extension of G that is Specker.

Proposition 2.7. If B is not an atomic Boolean algebra that satisfies the CCC then the lateral completion of the Specker space generated by B is not essential closed.

Proof: Since B is not atomic there is a $x \in B$ such that no atom is below x. We can have a decreasing sequence $u > x > x_1 > \ldots > 0$ of B.

Consider the subset C = {u, x, x_1, …} as a linearly ordered subset of B, take the completion C^ of C then C^ satisfies the following conditions:

1) C^ has countable dense subordering
2) C^ is complete
3) C^ has no first or last element

Hence C^ is order isomorphic to R. So there is an isomorphic map φ from [0, 1) onto C^ such that if $r < s$ in R then $\varphi(r) < \varphi(s)$ in C^.
Define $g_r = r\varphi(r) \in [B]$^ for each $r \in [0, 1)$, then { g_r | $r \in [0, 1)$] does exist in [B]^. But $g \notin [B]^L$ since every element in $[B]^L$ only has countable range. Therefore $[B]^e \neq [B]^L$.

Corollary 2.8. Let G be the set of all periodic sequences of real numbers, then $G^e \neq G^L$.
Proof: Let B be the set of all periodic sequences of {0, 1} then B is a countable atomless Boolean algebra and hence a free Boolean algebra. Consequently, B satisfies the CCC. Since G = [B], by the proposition 2.7, $G^e \neq G^L$.

Theorem 2.9. Let G = [X] be a Specker group generated by the generalized booleab algebra X and R be the ring of real numbers then $R \otimes_Z G$ is isomorphic to the Specker space generated by X.

Proof: Let F be the Specker space generated by X, i.e. for all f ∈ F, f can be written as $\sum_{i=1}^{n} r_i x_i$ where $r_i \in R$ and $x_i \in X$ such that if $i \neq j$ then $x_i \wedge x_j = 0$.

Define α: RxG → F by α : (r,g) →rg = r $\sum_{i=1}^{n} r_i x_i$

Then α(r+s,g) = α(r,g) +α(s,g) and α(r, g+h) = α(r,g) + α(r,h).

Therefore, there is α^ : $R \otimes_Z G \rightarrow F$ such that α^(r⊗g) = rg.

Define β: F → $R \otimes_Z G$ by β($\sum_{i=1}^{n} r_i x_i$) = $\sum_{i=1}^{n} r_i \otimes x_i \in R \otimes_Z G$. Then

(1) B is well-defined

Proof: Suppose f has two representations: f = $\sum_{i=1}^{m} s_i y_i$ and f = $\sum_{i=1}^{n} r_i x_i$ we want to show β($\sum_{i=1}^{m} s_i y_i$) = β($\sum_{i=1}^{n} r_i x_i$).

Notice that

(a) $\vee x_i = \vee y_i$ so for all I = 1, 2, …n, $x_i = \vee_j (x_i \wedge y_j) = \sum_{j=1}^{m} x_i \wedge y_j$ and for all j = 1, 2, …m, $y_j = \sum_{i=1}^{n} x_i \wedge y_j$

(b) If $x_i \wedge y_j \neq 0$ then $s_j = r_i$.

$\beta(\sum_{i=1}^{n} r_i x_i) = \beta(r_1(\sum_{j=1}^{m} x_1 \wedge y_j) + \ldots + r_n(\sum_{j=1}^{m} x_n \wedge y_j))$

$= r_1 \otimes (\sum_{j=1}^{m} x_1 \wedge y_j) + \ldots + r_n \otimes (\sum_{j=1}^{m} x_n \wedge y_j))$

$= s_1 \otimes (\sum_{i=1}^{n} x_i \wedge y_1) + \ldots + s_m \otimes (\sum_{i=1}^{n} x_i \wedge y_m)$

$=\beta(s_1(\sum_{i=1}^{n} x_i \wedge y_1) + \ldots + s_m(\sum_{i=1}^{n} x_i \wedge y_m))$

$= \beta(\sum_{i=1}^{m} s_i y_i)$

(2) B is a group homomorphism.

For if f = rx and g = sy and x and y are disjoint then

β(rx + sy) = r⊗x + s⊗y = β(rx) +β(sy). If x ∧ y ≠ 0, then let w = x - x ∧ y, u = x ∧ y, and v = y - x ∧ y then we have

β(rx + sy) = β(rw +(r+s)u + sv)

= r⊗w + (r+s)⊗u +s⊗v = β(rx) + β(sy).

In general, if g = $\sum_{i=1}^{m} s_i y_i$ and f = $\sum_{i=1}^{n} r_i x_i$ then the set z_ks obtained by disjoinfying xis and yjs, are a finite disjoint subsets of X and we can write f = $\sum_{i=1}^{t} r_i z_i$ and g = $\sum_{i=1}^{t} s_i z_i$. Then β(f + g) =β($\sum_{i=1}^{t} (r_i + s_i) z_i$)

= $\sum_{i=1}^{t} r_i \otimes z_i$ + $\sum_{i=1}^{t} s_i \otimes z_i$

=β(f) + β(g)

Also α^β($\sum_{i=1}^{n} r_i x_i$) = α^($\sum_{i=1}^{n} r_i \otimes x_i$) = $\sum_{i=1}^{n} \alpha \wedge (r_i \otimes x_i)$ = $\sum_{i=1}^{n} r_i x_i$

βα^(r⊗g) = β(rg) = r⊗g. So α^ is a one to one homormorphism from R⊗G onto F as a Z-modules. Therefore $R\otimes_Z G$ ≅ F as vector spaces.

Now define f = $\sum_{i=1}^{n} r_i x_i \geq 0$ if $r_i \geq 0$ for all I = 1, 2, …, n. Then $R\otimes_Z G$ is a vector lattice since in the representation of f all x_i's are pairwise disjoint. Clearly the map α preserves the order. Therefore $R\otimes_Z G$ ≅ F as vector lattices.

Corollary 2.10. G =[X] is a Specker group then the Specker space G^s generated by X is isomorphic onto $R \otimes G$

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