Distribution of Order Parameter for Kuramoto Model

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Abstract

The synchronization in large populations of interacting oscillators has been observed abundantly in nature, emergining in fields such as physical, biological and chemical system. For this reason, many scientists are seeking to understand the underlying mechansim of the generation of synchronous patterns in oscillatory system. The synchronization is analyzed in one of the most representative models of coupled phase oscillators, the Kuramoto model. The Kuramoto model can be used to understand the emergence of synchronization in nextworks of coupled, nonlinear oscillators. In particular, this model presents a phase transition from incoherence to synchronization. In this paper, we investigated the distribution of order parameter γ which describes the strength of synchrony of these oscillators. The larger the order parameter γ is, the more extent the oscillators are synchronized together. This order parameter γ is a critical parameter in the Kuramoto model. Kuramoto gave a initial estimate equation for the value of the order parameter by giving the value of the coupling constant. But our numerical results show that the distribution of the order parameter is slightly different from Kuramoto's estimation. We gave an estimation for the distribution of order parameter for different values of initial conditions. We discussed how the numerical result will be distributed around Kuramoto's analytical equation.

Introduction

In the past decades, the synchronization in complex networks has been a research topic in many areas of physics, biology and engineering [References]. Synchronization is the process by which interacting, oscillating objects affect each others phases that they spontaneously lock to common frequency. In other words, each oscillators frequency has locked onto the same values as all the other oscillators [References]. Synchronization has been observed in many real-world systems, such as networks of pacemaker cells in the heart [2, 43], circadian pacemaker cells in the suprachiasmatic nucleus of the brain [3] and stellate cells in the entorhinal cortex layer II of the brain [1], large populations of fireflies [4], superconductors [11], laser light [44] and microwave oscillators [6].

Synchronization always related to ”rhythm”, and it also means some interaction or coupling of oscillating systems. If we have an active oscillating system, it contains an internal source of energy that is transformed into oscillatory movement. The oscillators continue to generate the same rhythm until the source of energy expires. In 2- oscillators system, There are two basic types of synchronization: Anti-phase and in-phase synchronization. In both cases, an oscillator is locked in a fixed position comparing to the other one, and both of the oscillators are moving in a rotating frame. In a large oscillators system, if a number of oscillators are gathered together, we usually consider one cluster to be one single oscillator. There is one article describe synchronization problems [33], they address mechanical oscillators and dynamical systems. Another precise and mathematical introduction to the emergence of synchronization can be found in [13, **?**], which include some interesting synchronization like fireflies flashing together. We can find an introduction to the synchronization theory illustrated by various biological examples like transcranial brain stimulation given in [34], [15] [46]. A theory of synchronization of self-sustained oscillators was developed Stratonovich[9]. The influence of noise on mutual synchronization of two oscillators and the effect of fluctuating parameters are described by A. N. Malakhov [10].

Among many models that have been proposed to address synchronization phenomena, one of the most successful models is the Kuramoto model [18]. Y.Kuramoto developed this model in phase approximation that allows a description of globally coupled oscillators. This model can be used to understand the generation of synchronization in large networks of coupled, nonlinear oscillators. it also represent a synchronization in large populations and synchronization of distributed systems. In particular, this model presents a second-order phase transition from incoherence to synchronization. Kuramoto model describes a phase interaction which generates some interesting synchronization phenomena for globally weak coupling of large number of nonlinear oscillators. Kuramoto found that there is a certain value of the coupling constant, K_C, we call it a critical value, above which synchronization can occur, and below which it cannot. For any distribution of the natural frequencies of the oscillators, he was able to calculate K_C. For example, for a Lorentzian distribution of natural frequencies, K_C is just equal to the full width at half-max of the Lorentzian curve. For other distributions, the formula for K_C is more complex, but we can still calculate it [8].

Kuramoto model

In the 1960s, scientists began to build mathematical models for synchronization in many natural systems. Particularly, Arthur Winfree's model become very popular. He gave a model in which each oscillator's phase is determined by combining the state of all of the oscillators. In his model, the rate of change of the phase of an oscillator is determined by its own natural frequency ω_i and the state of all of the other oscillators combined. Each oscillator's sensitivity to the combination is represented as a function Y, and its own contribution to the combination is given by a function X. Then each oscillator has an equation to describe how its phase changes [12] [13] [14]:

$$\theta_i' = \omega_i + \sum_{j=1}^{N} X(\theta_j)Y(\theta_i) \qquad (1)$$

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Here θ_i is the phase of oscillator, $\dot{\theta}_i$ is the rate of change of phase of oscillator, ω_i is the natural frequency of oscillator i, and N is the total number of oscillators.

Winfree made numerical simulations and analytical approximations for this model and found that if the coupling is large enough, the oscillators could synchronize.

In 1975, Japanese scientist Yoshiki Kuramoto was inspired by Winfree's works, and he began exploring the behavior of collective synchronization. He used the following assumptions:

1. The oscillators are almost identical.
2. The coupling among oscillators is small.

After some complicated mathematical averaging, he proved that long term dynamics of any system of almost identical, weakly coupled limit cycle oscillators system have the following govern equation [8, 13]:

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}(\theta_j - \theta_i) \qquad (2)$$

Here the interaction function Γ_{ij} determines the form of coupling between oscillator i and oscillator j.

Kuramoto assumed that each oscillator take part in the affecting other oscillators. He called the interaction ”global coupling”.

He further assumed that the coupling were equally weighted can be expressed by a sin function of the difference of phases.

$$\Gamma_{ij}(\theta_j - \theta_i) = \frac{K}{N}\sin(\theta_j - \theta_i) \tag{3}$$

This derives the govern equations for Kuramoto model:

$$\theta_i' = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) \tag{4}$$

Here K is the coupling constant, and N is the total number of oscillators. The model assumed that N is very large, i.e large number of oscillators. The natural frequencies ω_i distributed by a probability density function $g(\omega)$, and it is symmetric about some value Π: $g(\Pi+\omega) = g(\Pi-\omega)$.
To simplify the governing equation of Kuramoto model, we need to define the order parameter, the order parameter describes the "mean field of the system".
Let us write the governing equations of Kuramoto model in terms of order parameter:

$$re^{i\psi} = \frac{1}{N}\Sigma_{j=1}^{N} e^{i\theta_j} \tag{5}$$

Here ψ is the average phase of all the oscillators.
The order parameter γ is distributed between 0 and 1. When $\gamma = 1$, we say that the oscillators are asynchronous, it also means their phases are completely spread around the unit circle. The synchrony measure γ is also called the phase coherence.

The formulations of the Relations between the Order Parameter *r* and the Critical Point K_c

The modulus of *r*, is a measure of the coherence of the oscillator system, it describes how close the oscillators are together. If we increase the order parameter, the phases of the oscillators will get closer together. The graphs in Fig. 1 show the order parameter being an arrow pointing from the center of the circle.
Let us consider (5): Multiplying both sides by $e^{-i\theta i}$ we get:

$$re^{i(\psi-\theta_i)} = \frac{1}{N}\Sigma_{j=1}^{N} e^{i(\theta_j-\theta_i)} \tag{6}$$

$$r\sin(\psi - \theta_i) = \frac{1}{N}\Sigma_{j=1}^{N}\sin(\theta_j - \theta_i) \tag{7}$$

Therefore, Equation (4) may be rewritten as

(8)

$$\dot{\theta}_i = \omega_i + Kr\sin(\psi - \theta_i) \text{ for } i = 1,2,...N.$$

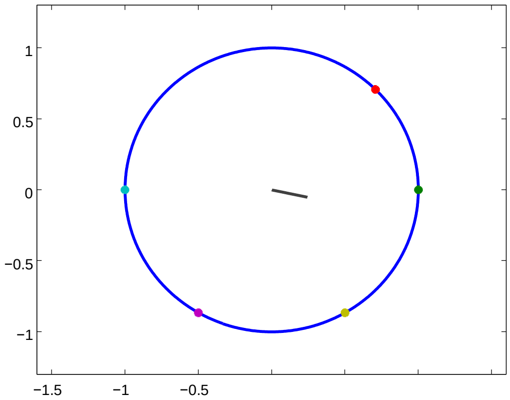
The corresponding stationary density:

$$\rho = \begin{cases} \delta[\theta - \psi - \sin^{-1}(\frac{w}{Kr})]\ H(\cos\theta) & when\ |\omega| < Kr \\ \dfrac{C}{|\omega - Kr\sin(\theta-\psi)|} & when\ |\omega| \geq Kr \end{cases}$$

4 (9)

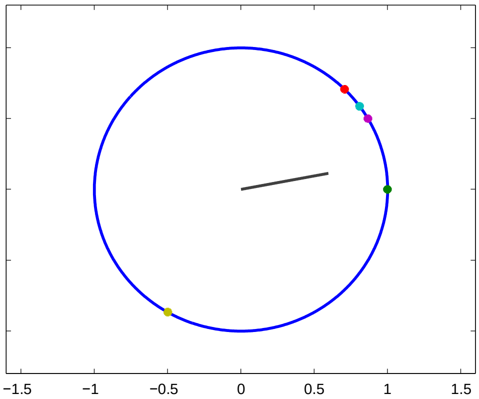
A

r= 0.2490 r= 0.6056



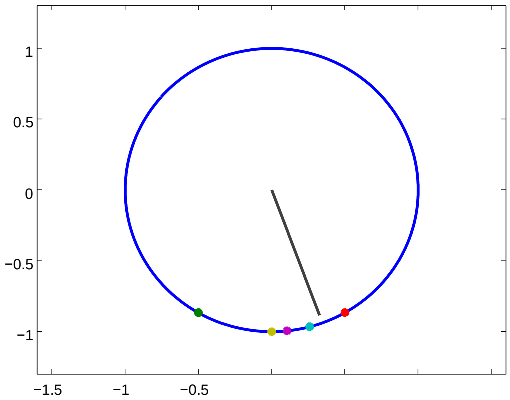
1
0.5
0
−0.5
−1

0 0.5 1 1.5



B

r= 0.9413 r= 0.9907



1
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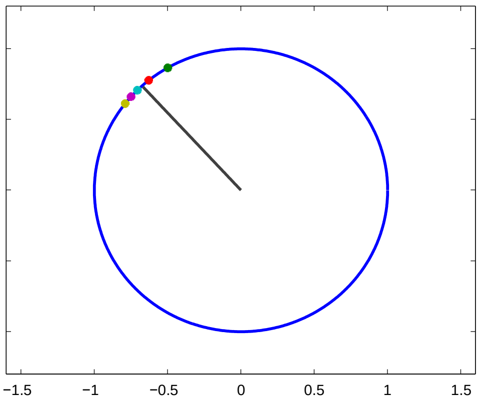


Figure 1: The order parameter is represented by the vector pointing from the center of the unit circle.

Here $H(x)$ is the Heaviside unit step function.

We take the natural frequency density function to be the Lorentzian density, defined as

$$g(\omega) = \frac{\gamma}{\pi(\gamma^2 + \omega^2)} \tag{10}$$

Actually $r(t)$ does not depend on time or $\psi(t)$ and $\psi(t)$ rotates uniformly at an angular frequency ϕ. We can set up a frame of reference that is rotating at the same frequency. Hence r(t) is stationary. So we can set $\psi(t)$ to any constant value. Without loss of generality, set $\psi(t) \equiv 0$ in the rotating frame. So we get

$$\dot{\theta}_i = \omega_i - Kr\sin\theta_i \tag{11}$$

and correspondingly, the stationary density function is

$$\rho = \begin{cases} \delta[\theta - \sin^{-1}(\frac{w}{Kr})]\ H(\cos\theta), & when\ |\omega| < Kr \\ \dfrac{C}{|\omega - Kr\sin\theta|} & ,when\ |\omega| \geq Kr \end{cases} \tag{12}$$

then

$$\begin{aligned} re^{i\psi} &= re^{i0} = r \\ &= \frac{1}{N}\Sigma_{j=1}^{N} e^{i\theta_j} = < e^{i\theta} > \\ &= < e^{i\theta} >_{lock} + < e^{i\theta} >_{unlock} \end{aligned}$$

According to (12)

$$\rho(\theta + \pi, -\omega) = \rho(\theta,\omega)$$

Compute the contribution of unlocked oscillators:

$$< e^{i\theta} >_{unlock} = \int_{-\pi}^{\pi}\int_{-\infty}^{-Kr} e^{i\theta}\rho(\theta,\omega)g(\omega)\ d\omega\ d\theta + \int_{-\pi}^{\pi}\int_{Kr}^{\infty} e^{i\theta}\rho(\theta,\omega)g(\omega)\ d\omega\ d\theta = I_1 + I_2 \tag{13}$$

where

$$I_1 = \int_{-\pi}^{\pi}\int_{-\infty}^{-Kr} e^{i\theta}\rho(\theta,\omega)g(\omega)\ d\omega\ d\theta = -\int_{-\pi}^{\pi}\int_{\infty}^{Kr} e^{i\theta}\rho(\theta,-\omega)g(-\omega)\ d\omega\ d\theta$$

Let $\theta' = \theta - \pi$, then

$$I_1 = -\int_{-2\pi}^{0}\int_{\infty}^{Kr} e^{i\theta'}e^{i\pi}\rho(\theta' + \pi, -\omega)g(-\omega)\ d\omega\ d\theta'$$

Since $\rho(\theta' + \pi, -\omega) = \rho(\theta',\omega)$, $g(-w) = g(w)$, $e^{i\pi} = -1$, we hence have

$$I_1 = -\int_{-2\pi}^{0}\int_{Kr}^{\infty} e^{i\theta'}e^{i\pi}\rho(\theta',\omega)g(\omega)\ d\omega\ d\theta'$$

Because the periodic boundary condition, we can shift θ' interval with any constant. So we can shift the integral interval to right with π.

So

$$I_1 = -\int_{-\pi}^{\pi}\int_{Kr}^{\infty} e^{i\theta'}e^{i\pi}\rho(\theta',\omega)g(\omega)\ d\omega\ d\theta' = -I_2 \tag{14}$$

According to (13),

$$< e^{i\theta} >_{unlock} = I1 + I2 = 0$$

So the unlocked oscillators have no contributions.

The locked oscillators are centered symmetrically on 0, therefore $< \sin\theta >_{lock} = 0$ and

r ∫ $=< ei\theta >lock$ $= < \cos\theta >lock$

Kr

$=cos(\theta(\omega))g(\omega)d\omega - Kr$

Consider (11) and (12)

$$r = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta g(Kr\sin\theta) Kr\cos\theta \; d\theta$$

$$= Kr \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta g(Kr\sin\theta) \; d\theta \qquad (15)$$

This implies

$$1 = K \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta g(Kr\sin\theta) \; d\theta$$

When make $r \to 0^+$ in the above equation, we can find the critical point K_c at which the order parameter rises from zero.

$$1 = K_c \int_{-\pi/2}^{\pi/2} \cos^2\theta g(0) \; d\theta$$

$$= K_c g(0) \int_{\pi/2}^{\pi/2} \cos^2\theta \; d\theta \;\; = \;\; K_c g(0)\frac{\pi}{2}$$

Hence

$$K_c = \frac{2}{\pi g(0)} \qquad (16)$$

Plug in (10): the function of $g(w)$

$$1 = K \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta \frac{\gamma}{\pi(\gamma^2 + K^2 r^2 \sin^2\theta)} \; d\theta$$

$$= \frac{K\gamma}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1-\sin^2\theta}{\gamma^2 + K^2 r^2 \sin^2\theta} \; d\theta$$

$$= -\frac{\gamma}{Kr^2} + \frac{Kr}{\pi}(1 + \frac{\gamma^2}{K^2r^2}) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\gamma^2 + K^2 r^2 \sin^2\theta}$$

$$= -\frac{\gamma}{Kr^2} + 2\frac{Kr}{\pi}(1 + \frac{\gamma^2}{K^2r^2}) \int_0^{\infty} \frac{d(\tan\theta)}{\gamma^2 \sec^2\theta + K^2 r^2 \tan^2\theta}$$

$$= -\frac{\gamma}{Kr^2} + 2\frac{Kr}{\pi}(1 + \frac{\gamma^2}{K^2r^2}) \int_0^{\infty} \frac{du}{\gamma^2(1+u^2) + K^2 r^2 u^2}$$

$$= -\frac{\gamma}{Kr^2} + \frac{\sqrt{K^2r^2 + \gamma^2}}{Kr^2}$$

Therefore we have

$$Kr^2 = -\gamma + \sqrt{K2r2 + \gamma 2}$$

and hence

$$r = \sqrt{1 - \frac{2\gamma}{K}}. \qquad (17)$$

To make the process of synchronization clear, the graphs Fig. 2 shows how the order parameter r rises as the coupling K between oscillators is increased. Numerical curves are taken from 500 oscillators with natural frequencies distributed with Lorentzian distribution:

Numerical and Theoretical curve of phase synchronization

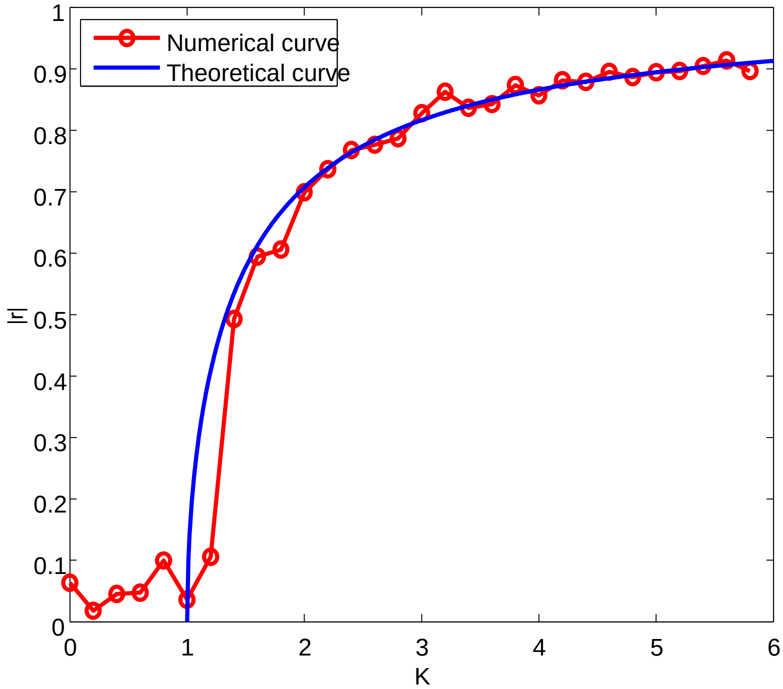


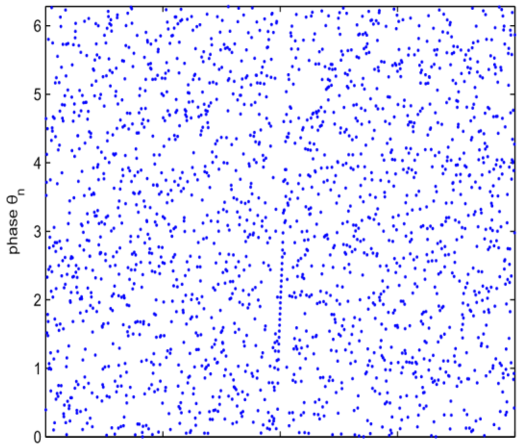
Figure 2: Numerical curve and analytical curve for order parameter by changing K

$$g(\omega) = \frac{\gamma}{\pi(\gamma^2 + \omega^2)}$$

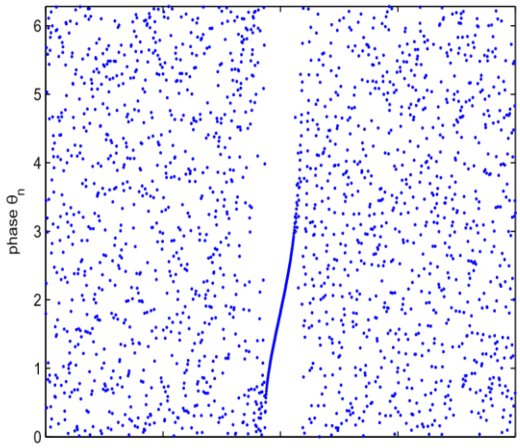
with $\gamma = 0.5$. Analytical curve is given by (17), according to (16), $K_c = 1$.

There is another way to visualize the synchronization: There are three graphs in Figure 3. The oscillators are numbered from the lowest to highest natural frequency, natural frequencies also distributed by Lorentzian distribution.

k=0.7; 2000 oscillators not synchronized



k=1; 2000 oscillators start to partially synchronized



0 500 1000 1500 2000 0 500 1000 1500 2000 oscillator n oscillator n

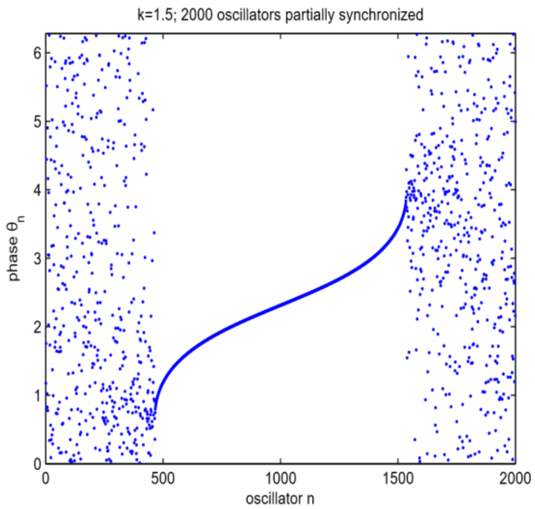


Figure 3: The synchronize phase θ_n of 2000 oscillators, take K=0.7, 1, 1.5

$$g(\omega) = \frac{\gamma}{\pi(\gamma^2 + \omega^2)}$$

with $\gamma = 0.5$.

So

$$K_c = \frac{2}{\pi g(0)} = 1$$

, We can see obvious partial synchronization at or above K_c.

Results

The Density Functions of Locked Terms and Unlocked Terms

To prove that the locked terms and unlocked terms are independent, we first try to get the probability density function of locked terms and unlocked terms separately.

Suppose the probability density function of locked terms is $f(y)$, then it satisfies the following equation:

$$P(y \le \cos\theta \le y + dy | \ \theta \ locked) = f(y)\ dy$$

$$\frac{P(\arccos(y+dy) \le \theta \le \arccos y)}{P(\theta\ locked)} = f(y)\ dy$$

For $y > 0$,

$$LHS = \frac{2P(\sin(\arccos(y+dy)) \le \frac{\omega}{kr} \le \sin(\arccos y))}{P(\theta\ locked)}$$

This derives

$$P(\theta\ locked) = \int_{-kr}^{kr} \frac{\gamma}{\pi(\gamma^2+\omega^2)} d\omega = \frac{2}{\pi}\arctan(\frac{kr}{\gamma})$$

$$P(\sin(\arccos(y+dy)) \le \frac{\omega}{kr} \le \sin(\arccos y)))$$
$$= P(kr\sin(\arccos(y+dy)) \le \omega \le kr\sin(\arccos y))$$
$$= P(kr\sqrt{1-(y+dy)^2} \le \omega \le kr\sqrt{1-y^2})$$
$$= \int_{kr\sqrt{1-(y+dy)^2}}^{kr\sqrt{1-y^2}} \frac{\gamma}{\pi(\gamma^2+\omega^2)} d\omega$$
$$= \frac{1}{\pi}\int_{\frac{kr}{\gamma}\sqrt{1-(y+dy)^2}}^{\frac{kr}{\gamma}\sqrt{1-y^2}} \frac{du}{1+u^2}$$

$$= \frac{1}{\pi}(\arctan(\frac{kr}{\gamma}\sqrt{1-y^2}) - \arctan(\frac{kr}{\gamma}\sqrt{1-(y+dy)^2}))$$

Let $\alpha = \arctan(\frac{kr}{\gamma}\sqrt{1-y^2}) - \arctan(\frac{kr}{\gamma}\sqrt{1-(y+dy)^2})$

As $dy \to 0$, $\alpha \to 0$, $\alpha \to \tan\alpha$

So

$$\alpha \to \tan\alpha = \frac{\frac{kr}{\gamma}(\sqrt{1-y^2}-\sqrt{1-(y+dy)^2})}{1+(\frac{kr}{\gamma})^2\sqrt{(1-y^2)(1-(y+dy)^2)}}$$

As $dy \to 0$,

$$\frac{k\ r\ y\ dy}{\gamma\sqrt{1-y^2}}\frac{1}{1+(kr/\gamma)^2(1-y^2)} \to \frac{k\ ry\ \gamma\ \sqrt{1-y^2}\ dy}{\gamma^2(1-y)^2+(kr(1-y^2))^2}$$

This derives the density function: for $0 \le y \le 1$,

$$f(y) = \frac{k\ r\ \gamma\ y\sqrt{1-y^2}}{arctan(\frac{kr}{\gamma})(\gamma^2(1-y^2)+(kr(1-y^2))^2)}$$

The density function of unlocked term satisfies:

$$P(y \le \cos\theta \le y + dy|\ \theta\ unlocked) = f^{-}(y)\ dy$$

For $0 \le y \le 1$, $y \le \cos\theta \le y + dy$.

So $\sqrt{1-(y+dy)^2} \le \sin\theta \le \sqrt{1-y^2},\ d\theta = \frac{dy}{\sqrt{1-y^2}}$

or $-\sqrt{1-y^2} \le \sin\theta \le -\sqrt{1-(y+dy)^2},\ d\theta = \frac{dy}{\sqrt{1-y^2}}$

$$\rho(\theta,\omega) = \frac{C}{|\theta'|} = \frac{C}{|\omega - kr\sin\theta|}$$

So

$$1 = \int_{-\pi}^{\pi}\rho(\theta,\omega)d\theta = C\int_{-\pi}^{\pi}\frac{d\theta}{|\omega - kr\sin\theta|}$$

Which derives

$$C = \frac{\sqrt{\omega^2-(kr)^2}}{2\pi}$$

So

$$P(y \le \cos\theta \le y+dy|\theta\ unlocked) = \frac{P(y \le \cos\theta \le y+dy, |\omega| > kr)}{P(|\omega| > kr)}$$

$$= \frac{P(-\sqrt{1-(y+dy)^2} \le \sin\theta \le -\sqrt{1-y^2}, |\omega| > kr)}{P(|\omega| > kr)}$$

$$+ \frac{P(-\sqrt{1-y^2} \le \sin\theta \le -\sqrt{1-(y+dy)^2}, |\omega| > kr)}{P(|\omega| > kr)}$$

$$= \frac{\int_I g(\omega)\int_{I_1}\rho(\theta,\omega)d\theta d\omega + \int_I g(\omega)\int_{I_2}\rho(\theta,\omega)d\theta d\omega}{2\int_{kr}^{\infty}\frac{\gamma}{\pi(\gamma^2+\omega^2)}d\omega}$$

$$A_1 = \int_I g(\omega)\int_{I_1}\rho(\theta,\omega)d\theta d\omega$$

$$= \frac{1}{2\pi}\int_I \frac{\gamma}{\pi(\gamma^2+\omega^2)}\int_{I_1}\frac{\sqrt{\omega^2-(kr)^2}}{|\omega - kr\sqrt{1-y^2}|}d\theta d\omega$$

$$= \frac{1}{2\pi^2}\int_I \frac{\gamma}{\gamma^2+\omega^2}\int_{I_1}\frac{\sqrt{\omega^2-(kr)^2}}{|\omega - kr\sqrt{1-y^2}|}\frac{dy}{\sqrt{1-y^2}}d\omega$$

As $dy \to 0$, $d\theta \to 0$, $I_1 \to 0$, $I_2 \to 0$

$$A_1 \to \frac{1}{2\pi^2}(\int_I \frac{\gamma}{\gamma^2+\omega^2}\frac{\sqrt{\omega^2-(kr)^2}}{\omega - kr\sqrt{1-y^2}}\frac{1}{\sqrt{1-y^2}}d\omega)dy$$

$$= \frac{1}{2\pi^2\sqrt{1-y^2}}\int_I \frac{\gamma}{\gamma^2+\omega^2}\frac{\sqrt{\omega^2-(kr)^2}}{|\omega - kr\sqrt{1-y^2}|}d\omega dy$$

$$A_2 = \int_I \int_{I_2} g(\omega)\rho(\theta,\omega)d\theta d\omega$$
$$= \frac{1}{2\pi^2}\int_I \int_{I_2} \frac{1}{\sqrt{1-y^2}} \frac{\gamma}{\gamma^2+\omega^2} \frac{\sqrt{\omega^2-(kr)^2}}{|\omega+kr\sqrt{1-y^2}|} dy d\omega$$
$$= (\frac{1}{2\pi^2\sqrt{1-y^2}}\int_I \frac{\gamma}{\gamma^2+\omega^2} \frac{\sqrt{\omega^2-(kr)^2}}{|\omega+kr\sqrt{1-y^2}|} d\omega) dy$$

This derives

$$A_1 + A_2 = \frac{1}{2\pi^2\sqrt{1-y^2}}(\int_I \frac{\gamma\sqrt{\omega^2-(kr)^2}}{\gamma^2+\omega^2}(\frac{1}{|\omega+kr\sqrt{1-y^2}|} + \frac{1}{|\omega-kr\sqrt{1-y^2}|}))d\omega dy$$

Similarly, for $-1 \le y < 0$,

$$A_1 + A_2 = \frac{1}{2\pi^2\sqrt{1-y^2}}\int_I \frac{\gamma\sqrt{\omega^2-(kr)^2}}{\gamma^2+\omega^2}(\frac{1}{|\omega+kr\sqrt{1-y^2}|} + \frac{1}{|\omega-kr\sqrt{1-y^2}|})d\omega dy$$
$$= \frac{\gamma}{2\pi^2\sqrt{1-y^2}}\int_I \frac{\sqrt{\omega^2-(kr)^2}}{\gamma^2+\omega^2} \frac{2|\omega|}{\omega^2-(kr)^2(1-y^2)} d\omega dy$$
$$= \frac{2\gamma}{\pi^2\sqrt{1-y^2}}\int_{kr}^{\infty} \frac{\omega\sqrt{\omega^2-(kr)^2}}{(\gamma^2+\omega^2)(\omega^2-(kr)^2(1-y^2))} d\omega dy$$
$$= \frac{\gamma}{\pi^2\sqrt{1-y^2}}\int_{\alpha}^{\infty} \frac{\sqrt{t-\alpha}}{(t+\gamma^2)(t-\alpha(1-y^2))} dt dy$$
$$= \frac{\gamma}{\pi^2\sqrt{1-y^2}}\int_{0}^{\infty} \frac{\sqrt{t}}{(t+\alpha+\gamma^2)(t+\alpha y^2)} dt dy$$

$\Rightarrow$

$$\frac{P(y \le \cos\theta \le y+dy, |\omega| > kr)}{P(|\omega| > kr)} = \bar{f}(y)dy$$
$$= \frac{\gamma}{\pi^2\sqrt{1-y^2}} \frac{1}{1-\frac{2}{\pi}arctan(\frac{kr}{\gamma})}\int_0^{\infty} \frac{\sqrt{t}dtdy}{(t+\alpha+\gamma^2)(t+\alpha y^2)}$$
$$= \frac{\gamma}{\pi\sqrt{1-y^2}} \frac{1}{\pi-2arctan(\frac{kr}{\gamma})}\int_0^{\infty} \frac{\sqrt{t}dtdy}{(t+\alpha+\gamma^2)(t+\alpha y^2)}$$

$\Rightarrow$

$$\bar{f}(y) = \frac{\gamma}{\pi\sqrt{1-y^2}(\pi-2\arctan(kr/\gamma))}\int_0^{\infty} \frac{\sqrt{t}dt}{(t+\alpha+\gamma^2)(t+\alpha y^2)}$$
$$= \frac{\gamma}{\sqrt{1-y^2}(\pi-2arctan(kr/\gamma))} \frac{1}{\sqrt{\alpha+\gamma^2}+\sqrt{\alpha y^2}}$$
$$= \frac{\gamma}{\sqrt{1-y^2}(\pi-2arctan(kr/\gamma))} \frac{1}{\sqrt{(kr)^2+\gamma^2}+kr|y|}$$

So we have the density function f for locked part and $\bar{f}$ for unlocked part:

$$f(y) = \frac{k\ r\ \gamma\ y\sqrt{1-y^2}}{arctan(\frac{kr}{\gamma})(\gamma^2(1-y^2)+(kr(1-y^2))^2)} \quad (18)$$

$$\bar{f}(y) = \frac{\gamma}{\sqrt{1-y^2}(\pi-2arctan(kr/\gamma))(\sqrt{(kr)^2+\gamma^2}+kr|y|)} \quad (19)$$

To prove the locked terms are independent, we want to show that the sum of any two locked terms satisfies the analytical probability density function derived by convolution law:

If X and Y are two locked terms, Z is the sum of X and Y:

$$Z = X + Y$$

$$E(e^{tZ}) = \int_{-\infty}^{\infty} e^{tu} f_z(u)du \quad (20)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{tx} e^{ty} f_x(x)f_y(y)dxdu \quad (21)$$

$$= \int_{-\infty}^{\infty} e^{tu} \int_{-\infty}^{\infty} f_x(x)f_y(u-x)dxdu \quad (22)$$

$\Rightarrow$

$$f_z(u) = \int_{-\infty}^{\infty} f_x(x) f_y(u-x)dx$$

Similarly, to prove the unlocked terms are independent, we can also get the analytical probability density function of two unlocked terms by the convolution law:

If $\bar{X}$ and $\bar{Y}$ are two unlocked terms, $\bar{Z}$ is the sum of $\bar{X}$ and $\bar{Y}$:

$$\bar{Z} = \bar{X} + \bar{Y}$$

$$E(e^{t\bar{z}}) = \int_{-\infty}^{\infty} e^{tu} f_{\bar{z}}(u)du \quad (23)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t\bar{x}} e^{t\bar{y}} f_{\bar{x}}(\bar{x})f_{\bar{y}}(\bar{y})d\bar{x}d\bar{u} \quad (24)$$

$$= \int_{-\infty}^{\infty} e^{tu} \int_{-\infty}^{\infty} f_{\bar{x}}(\bar{x})f_{\bar{y}}(u-\bar{x})d\bar{x}d\bar{u} \quad (25)$$

$\Rightarrow$

$$f_{\bar{z}}(u) = \int_{-\infty}^{\infty} f_{\bar{x}}(\bar{x}) f_{\bar{y}}(u-\bar{x})d\bar{x}$$

In the following figures, we compare the numerical and analytical result for the density functions, the numerical result of density function for one oscillator satisfies the analytical result very well. The analytical density function of the sum of two locked oscillators and two unlocked oscillators are given by convolution law separately, and we also compare the results, the numerical curve also approximate the analytical curve very well. This shows the locked oscillators are independent and unlocked terms are also independent.

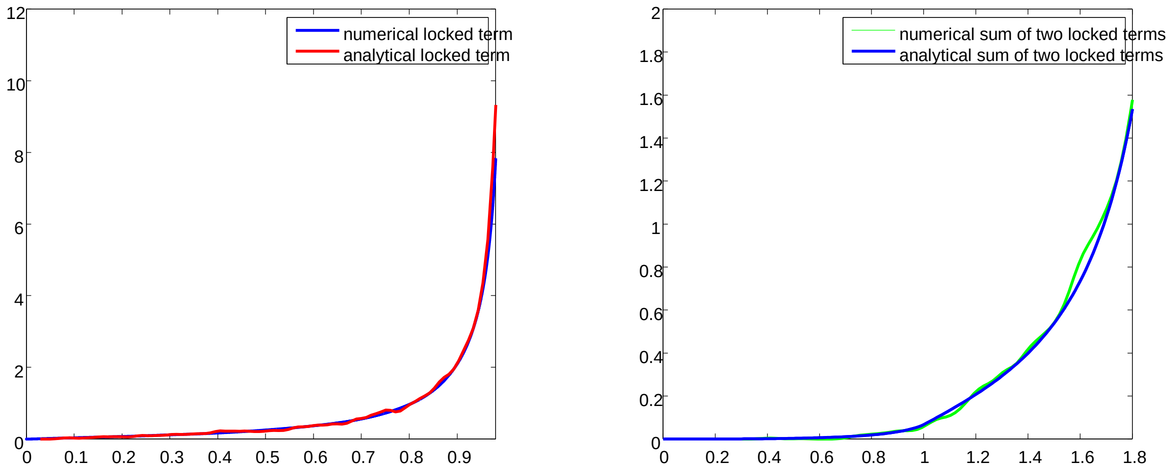


Figure 4: Comparison of the numerical and analytical density function for the locked oscillators and the sum of two locked oscillators

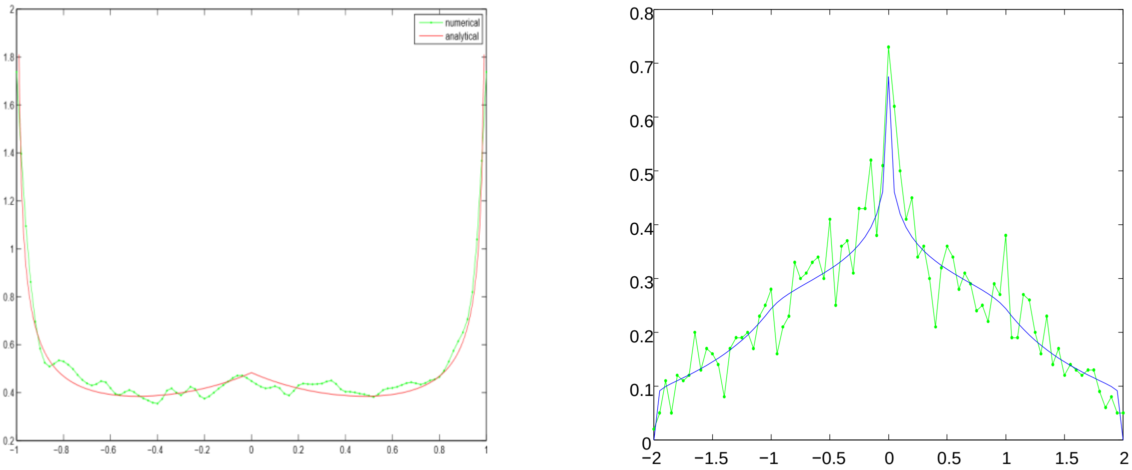


Figure 5: Comparison of the numerical and analytical density function for the unlocked oscillators and the sum of two unlocked oscillators

The Distribution of Order Parameter r

If we set $\psi \equiv 0$ (in a rotating frame), then $\sum_{j=1}^{N} \sin\theta_j = 0.$

The order parameter satisfies the following equation:

$$r = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j} = \frac{1}{N}\sum_{j=1}^{N} \cos\theta_j \quad . \qquad (26)$$

According to the analysis on the drift term, we know that the contribution of the drift term is 0. Namely

$$r = \langle \cos\theta \rangle_{lock} + \langle \cos\theta \rangle_{unlock}$$

$\approx \langle \cos\theta \rangle_{lock}$

There are totally N terms $\theta_1, \theta_2, \ldots \theta_N$. Without loss of generality, we suppose the first n terms $\theta_1, \theta_2, \ldots \theta_n$ are synchronized, and the last $N - n$ terms $\theta_{n+1}, \theta_{n+2}, \ldots \theta_N$ are not synchronized. Here

$0 < n < N$.

From equation (26), we get

$$\begin{aligned} r &= \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j} = \frac{1}{N}\sum_{j=1}^{N} \cos\theta_j \\ &= \frac{1}{N}\sum_{j=1}^{n} \cos\theta_j + \frac{1}{N}\sum_{j=n+1}^{N} \cos\theta_j \end{aligned} \qquad (27)$$

Suppose $S_1 = \sum_{j=1}^{n} \cos\theta_j, \;\; S_2 = \sum_{j=n+1}^{N} \cos\theta_j, \;\; S = \sum_{j=1}^{N} \cos\theta_j$,

then

$$S_1 + S_2 = S = Nr$$

and

$$S_2 \approx 0, \; S_1 \approx S = Nr$$

.

For a fixed number N, if we also fix a series of natural frequency $\omega_1, \omega_2, \ldots \omega_N$, and these series of natural frequency satisfies the Lorentzian distribution very well-proportionally, the density function is

$$g(\omega) = \frac{\nu}{\pi(\nu^2 + \omega^2)}$$

.

Take different series of initial values of θ_j, $j = 1,2,\ldots N$.

These series of θ_j are taken randomly which satisfy uniformly distribution. We will get different values of order parameter r for each series of θ_j, $j = 1,2,\ldots N$.

Question: When the value of N is very large, what is the distribution of the value of order parameter r?

When N is large, n and N-n are large. According to central limit theorem,

$$\frac{S_1 - n\mu_1}{\sigma_1\sqrt{n}} \sim N(0,1)$$

. Here μ_1 and σ_1 are the mean and variance of locked terms $\cos\theta_j$, $j = 1,2,...n$.

$$\frac{S_2 - (N-n)\mu_2}{\sigma_2\sqrt{N-n}} \sim N(0,1)$$

Here μ_2 and σ_2 are the mean and variance of unlocked terms $\cos\theta_j$, $j = n+1, n+2, ...N$

We know that S_1 and S_2 both satisfy Gaussian distribution. From equation (27) the order parameter

$$r = \frac{1}{N}(S_1 + S_2)$$

is the linear combination of Gaussian distributed functions. So r also satisfies Gaussian distribution. The variance of S_1 is $\sigma_1^2 n$ and the variance of S_2 is $\sigma_2^2(N-n)$, σ satisfies the following equation:

$$\sigma = \frac{1}{N^2}(\sigma_1^2 n + \sigma_2^2(N-n))$$

.

So the question is reduced to calculate the value of σ_1 and σ_2. σ_1 is the variance of locked terms $\cos\theta_1, \cos\theta_2, ..., \cos\theta_n$. $\sigma 1 = Var(\theta) locked = E(\cos 2\ \theta) locked - (E(\cos\theta) locked) 2$

$$E(\cos^2\theta)_{locked} = \frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta g(Kr\sin\theta)Kr\cos\theta\ d\theta}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(Kr\sin\theta)Kr\cos\theta\ d\theta} \tag{28}$$

Here

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta g(Kr\sin\theta)Kr\cos\theta\ d\theta = \frac{Kr\nu}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^3\theta}{\nu^2 + K^2r^2\sin^2\theta}\ d\theta$$
$$= \frac{2\nu}{\pi Kr}\left(\frac{\nu^2 + K^2r^2}{Kr\nu}\arctan\left(\frac{Kr}{\nu}\right) - 1\right)$$
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(Kr\sin\theta)Kr\cos\theta\ d\theta = \frac{Kr\nu}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta}{\nu^2 + K^2r^2\sin^2\theta}\ d\theta$$
$$= \frac{2}{\pi}\arctan\left(\frac{Kr}{\nu}\right)$$

Plug these results into the equation (28), we can get

$$E(\cos^2\theta)_{locked} = \frac{\nu}{Kr}\left(\frac{\nu^2 + K^2r^2}{Kr\nu} - \frac{1}{\arctan(\frac{Kr}{\nu})}\right) \tag{29}$$

The variance of unlocked terms is

$\sigma 2 = Var(\theta) unlocked = E(\cos 2\ \theta) unlocked - (E(\cos\theta) unlocked) 2$

$$E(\cos^2\theta)_{unlocked} = \int_{-\pi/2}^{\pi/2}\int_{|\omega|>Kr} \cos^2\theta \frac{Cg(\omega)}{|\omega - Kr\sin\theta|}\ d\theta\ d\omega$$
$$= 2\int_{-\pi/2}^{\pi/2}\int_{\omega>Kr} \cos^2\theta \frac{Cg(\omega)}{|\omega - Kr\sin\theta|}\ d\theta\ d\omega$$

and

$$E(\cos\theta)_{unlocked} = \int_{-\pi/2}^{\pi/2}\int_{|\omega|>Kr} \cos\theta \frac{Cg(\omega)}{|\omega - Kr\sin\theta|}\ d\theta\ d\omega$$
$$= 2\int_{-\pi/2}^{\pi/2}\int_{\omega>Kr} \cos\theta \frac{Cg(\omega)}{|\omega - Kr\sin\theta|}\ d\theta\ d\omega$$

Here $g(\omega) = \frac{\nu}{\pi(\nu^2+\omega^2)}$, $C = \frac{\sqrt{\omega^2 - K^2r^2}}{2\pi}$.

These integrals are relatively hard to simplify, but we can use numerical methods to get the variance of the unlocked terms. The variance of order parameter r is $\sigma = \frac{1}{N^2}(\sigma_1^2 n + \sigma_2^2(N-n))$.

i.e

$$\frac{r - mean(r)}{\frac{1}{N}\sqrt{\sigma_1^2 n + \sigma_2^2 (N-n)}} \sim N(0,1) \tag{30}$$

Taking 1000 series of uniformly distributed initial values $\theta_1, \theta_2, \ldots \theta_N$, we can get 1000 values of order parameter *r*. The order parameter *r* approximately satisfies Gaussian distribution, the equation for this distribution is (30). The graphs in Figure 6 compare the LHS of (30) with $N(0,1)$ (standard normal distribution). The blue line is the density function of standard normal distribution, the red line is the density function of the distribution of LHS function.

Conclusion

Kuramoto model is important for us to understand the dynamics of synchronization among genetic oscillators. Kuramoto already give a theoretical predict for the order parameter γ which describes

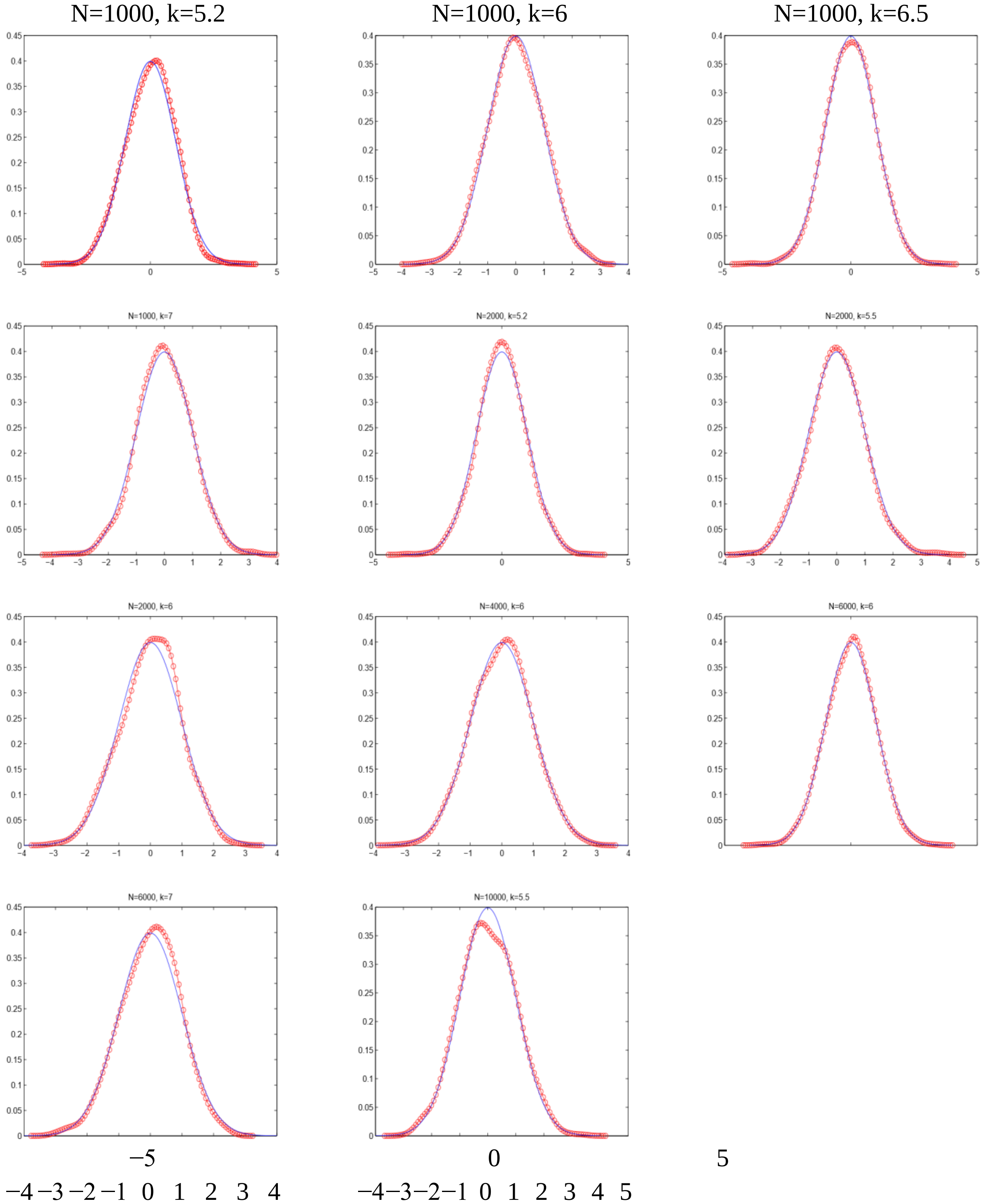


Figure 6: Comparison of distribution of order parameter and normal distribution

the strength the oscillators are coupled together. Our main purpose here is to further predict the numerical distribution around Kuramoto's theoretical prediction. To have a comprehensive understanding on the synchronization of all coupled oscillators, we investigated the density function for synchronized and unsynchronized oscillators individually. We found that when the number of oscillators is very large, in a rotating frame, the sin function of oscillators made no contribution to the order parameter. So we just need to consider cos function for these oscillators. We also found that the mechanism of the order parameters generated from these synchronized oscillators have their own distribution pattern. Basically, the distribution of numerical result of order parameter will satisfy a Gaussian distribution around Kuramoto's analytical prediction. When we make a comparison of the numerical result with the Gaussian distribution, Fig 6. shows that the numerical result basically coinside with a Gaussian distribution.

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