The Fascinating Double Angle Formulas of the Mulatu Numbers

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Abstract

The Mulatu numbers were introduced in [1]. The numbers are sequences of numbers of the form: **4,1, 5,6,11,17,28,45...** The numbers have wonderful and amazing properties and patterns.
In mathematical terms, the sequence of the Mulatu numbers is defined by the following recurrence relation:

$$M_n := \begin{cases} 4 & if\ n = 0; \\ 1 & if\ n = 1; \\ M_{n-1} + M_{n-2} & if\ n > 1. \end{cases}$$

The double Angel Formulas for Fibonacci and Lucas numbers are given by the following formulas respectively**.**

(1) $F_{2n} = F_n L_n$ **and** (2) $L_{2n} = \dfrac{5F^2{}_n + L^2{}_n}{2}$

Since both the Fibonacci and Lucas numbers have double angle Formulas, It is natural to ask if such formula exist for Mulatu Numbers. The answer is affirmative and produces the following paper.

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1. Introduction and Background.

As given in [1], the Mulatu numbers are a sequence of numbers recently introduced by Mulatu Lemma, an Ethiopian Mathematician and Professor of Mathematics at Savannah State University, Savannah, Georgia, USA. The Mulatu sequence has wealthy mathematical properties and patterns like the two celebrity sequences of Fibonacci and Lucas.

In this paper, more interesting relationships of the Mulatu numbers to the Fibonacci and Lucas numbers will be presented.

Here are the First 21 Mulatu, Fibonacci, and Lucas numbers for quick reference.

Mulatu(M_n)**, Fibonacci**(F_n) **and Lucas**(L_n) **Numbers**
(Tables 1 & 2)

Table 1

n:	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
M_n:	4	1	5	6	11	17	28	45	73	118	191	309
F_n:	0	1	1	2	3	5	8	13	21	34	55	89
L_n:	2	1	3	4	7	11	18	29	47	76	123	199

Table 2

n:	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
M_n	500	809	1309	2118	3427	5545	8972	14517	23489
F_n:	144	233	377	610	987	1597	2584	4181	6765
L_n:	322	521	843	1364	2207	3571	5778	9349	15127

Remark 1: Throughout this paper M, F, and L stand for Mulatu numbers, Fibonacci numbers, and Lucas number respectively.

The following well-known identities of Mulatu numbers, Fibonacci numbers, and Lucas numbers are required in this paper and hereby listed for quick reference.

(1) $L_n = F_{n-1} + F_{n+1}$

(2) $F_{n+1} = F_n + F_{n-1}$

(3) $F_{2n} = F_n L_n$

(4) $L_{2n} = F_n + 2F_{n-1}$

(5) $F_n = \dfrac{L_{n+1} + L_{n-1}}{5}$

(6) $L_{n+1} = L_n + L_{n-1}$

(7) $F_{n+k} = F_{n-1}F_k + F_n F_{k+1}$

(8) $5F^2{}_n - L^2{}_n = 4(-1)^{n+1}$

(9) $L_{n+m} = \dfrac{5F_n F_m + L_n L_m}{2}$

(10) $M_{n+k} = F_{n-1}M_k + M_{k+1}F_n$

The Main Results.

We will state the following theorem proved in [1] as proposition 1 and use it.

Proposition 1. $M_n = F_{n-3} + F_{n-1} + F_{n+2}$

Theorem 1: The following are equivalent.

(1) M_n

(2) $F_{n-3} + F_{n-1} + F_{n+2}$

(3) $L_n + 2F_{n-1}$

(4) $F_n + 4F_{n-1}$

(5) $4F_{n+1} - 3F_n$

Proof: We will show that (1)$\Rightarrow$ (2) $\Rightarrow$(3) $\Rightarrow$(4) $\Rightarrow$(5) $\Rightarrow$(1)

(i) (1)$\Rightarrow$ (2) follows by Proposition 1.

(ii) (2))$\Rightarrow$(3) follows as shown:

$$
\begin{aligned}
F_{n-3} + F_{n-1} + F_{n+2} &= F_{n-3} + F_{n-1} + F_{n+1} + F_n \\
&= F_{n-3} + F_{n-1} + F_{n+1} + F_{n-1} + F_{n-2} \\
&= F_{n-1} - F_{n-2} + F_{n-1} + F_{n+1} + F_{n-1} + F_{n-2} \\
&= 2F_{n-1} + L_n
\end{aligned}
$$

(iii)(3) $\Rightarrow$(4) follows as shown:

$$
\begin{aligned}
L_n + 2F_{n-1} &= F_{n+1} + F_{n-1} + 2F_{n-1} \\
&= F_n + F_{n-1} + F_{n-1} + 2F_{n-1} \\
&= F_n + 4F_{n-1}
\end{aligned}
$$

(iv) (4) $\Rightarrow$(5) follows as shown:

$$
\begin{aligned}
F_n + 4F_{n-1} &= F_n + 4(F_{n+1} - F_n) \\
&= 4F_{n+1} - 3F_n
\end{aligned}
$$

(v) (5) $\Rightarrow$(1) follows as shown:

$$4F_{n+1} - 3F_n = 4F_{n+1} - 3(F_{n+1} - F_{n-1}) = F_{n+1} + 3F_{n-1} = F_{n+1} + F_{n-1} + F_{n-1} + F_{n-1}$$

$$= F_{n+1} + (F_n - F_{n-2}) + F_{n-1} + F_{n-3} + F_{n-2} = F_{n+1} + F_n - F_{n-2} + F_{n-1} + F_{n-3} + F_{n-2}$$

$= F_{n+2} + F_{n-1} + F_{n-3} = M_n$ by Proposition 1 and hence) (5) $\Rightarrow$(1). Thus the theorem is proved.

Theorem 2: $L_n^2 = F_{n+1}(M_n + F_n) - F_{2n}$

Proof: $L_n^2 = (F_n + 2F_{n-1})^2 = F_n^2 + 4F_n F_{n-1} + 4F_{n-1}^2$

$$
\begin{aligned}
&= -F_n(F_n + 2F_{n-1}) + (F_n + F_{n-1})(F_n + 4F_{n-1}) + F_n^2 + F_n F_{n-1} \\
&= -F_n L_n + F_{n+1} M_n + F_n(F_n + F_{n-1}) \\
&= -F_n L_n + F_{n+1} M_n + F_{n+1} F_n) \\
&= -F_n L_n + F_{n+1}(M_n + F_n) \\
&= F_{n+1}(M_n + F_n) - F_n L_n \\
&= F_{n+1}(M_n + F_n) - F_{2n}
\end{aligned}
$$

Hence the theorem is proved.

Theorem 3. $M^2 = F_{2n} + 6F_{n-1}F_n + 16F^2{}_{n-1}$

Proof: $M^2 = MM = (L_n + 2F_{n-1})(F_n + 4F_{n-1})$

$= L_n F_n + 4F_{n-1}(F_n + 2F_{n-1}) + 2F_{n-1}F_n + 8F^2{}_{n-1}$

$= F_{2n} + 6F_{n-1}F_n + 8F^2{}_{n-1} + 8F^2{}_{n-1}$

$= F_{2n} + 6F_{n-1}F_n + 16F^2{}_{n-1}$

Hence, the theorem is proved.

Lemma 1. $F_{2n-1} = F^2{}_n + F^2{}_{n-1}$

Proof: Applying (7) above, we have $F_{2n-1} = F_{n+n-1} = F^2{}_{n-1} + F^2{}_n$

Lemma 2. $M_{n+1} = F_{n-1} + 5F_n$

Proof: using (10) above, we have

$M_{n+1} = F_{n-1}M_1 + F_nM_2 = F_{n-1} + 5F_n$.

Theorem 4. The following are equivalent.

1. M_{2n}
2. $F_{2n} + 4F_{2n-1}$
3. $4F_{2n+1} - 3F_{2n}$
4. $L_{2n} + 2F_{2n-1}$
5. $\dfrac{9F^2{}_n + L^2{}_n + 4F^2{}_{n-1}}{2}$
6. $M_nL_n + 5F^2{}_n - L^2{}_n$

Proof $(1)\Rightarrow(2)\Rightarrow(3)\Rightarrow(4)$ follows by Theorem 1. We will suffice to show that $(4)\Rightarrow(5)\Rightarrow(6)\Rightarrow(1)$

(i) $(4)\Rightarrow(5)$. Note that Using (9) above and lemma 1, we have $L_{2n} + 2F_{2n-1} = \dfrac{5F^2{}_n + L^2{}_n}{2} + 2F^2{}_n + 2F^2{}_{n-1}$

$= \dfrac{9F^2{}_n + L^2{}_n + 4F^2{}_{n-1}}{2}$.

(ii) $(5)\Rightarrow(6)$. We show this using Theorem 3 and Lemma1.

Note that

$$\frac{9F^2{}_n + L^2{}_n + 4F^2{}_{n-1}}{2} = \frac{5F^2{}_n + L^2{}_n}{2} + 2F^2{}_n + 2F^2{}_{n-1}$$

$$= L_{2n} + 2F_{2n-1} = F_{2n} + 2F_{2n-1} + 2F_{2n-1} = F_nL_n + 4F_{2n-1}$$

$$= F_n(F_n + 2F_{n-1}) + 4F^2{}_{n-1} + 4F^2{}_n = 5F^2{}_n + 4F^2{}_{n-1} + 2F_{n-1}F_n =$$

$$= (F^2{}_n + 8F^2{}_{n-1} + 6F_{n-1}F_n) + 5F^2{}_n - (F^2{}_n + 4F_{n-1}F_n + 4\mathrm{F}^2{}_{n-1})$$
$$= (F_n + 4F_{n-1})(F_n + 2F_{n-1}) + 5F^2{}_n - (F_n + 2F_{n-1})^2$$
$$= M_nL_n + 5F^2{}_n - L^2{}_n$$

(iii) (6)$\Rightarrow$(1). We show this using Theorem 3 and Lemma 2.

Note that

$$M_nL_n + 5F^2{}_n - L^2{}_n = M_nL_n - L^2{}_n + 5F^2{}_n$$
$$= M_nL_n - (F_n + 2F_{n-1})^2 + 5F^2{}_n$$
$$= M_nL_n - (F^2{}_n + 4F_{n-1}F_n + 4\mathrm{F}^2{}_{n-1}) + 5F^2{}_n$$
$$= M_nL_n - (F_n + F_{n-1})(F_n + 4F_{n-1}) + F_nF_{n-1} + 5F^2{}_n$$
$$= M_nL_n - F_{n+1}M_n + F_nF_{n-1} + 5F^2{}_n$$
$$= M_n(L_n - F_{n+1}) + F_n(F_{n-1} + 5F_n)$$
$$= M_nF_{n-1} + F_nM_{n+1}$$
$$= M_{n+n} = M_{2n}.$$

Hence, (6)$\Rightarrow$(1) and the theorem is proved.

Corollary 1: $M_{2n} = M_nL_n + + 4(-1)^{n+1}$

Proof: The corollary easily follows by Theorem 4, using (8) above.

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References

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