A Note on The Abel Matrix Transformations

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Abstract

Let t be sequence in (0,1) that converges to 1. The Abel matrix is defined as a_{nk}= (1-t_n)^kt_n. We denote the Abel Matrix by A_t. A_t is a sequence to sequence mapping? When a matrix A_t is applied to a sequence x, we get a new sequence $A_t x$ whose nth term is given by:

$$(A_t x)_n = (1 - t_n) \sum_{k=o}^{\infty} t_n{}^k x_k$$

The sequence $A_t x$ is called the A_t -transform of the sequence x.

The purpose of this research is to investigate the effect of applying A_t to convergent sequences, bounded sequences, divergent sequences, and absolutely convergent sequences. We considering and answer the following interesting main research questions.

Research Questions.

(1) What is the domain of t for which A_t maps convergent sequence into convergent sequence?

(2) What is the domain of t for which the A_t maps absolutely convergent sequence into absolutely convergent sequence?

(3) Does A_t maps unbounded sequence to convergent sequence?

(4) Does A_t maps divergent sequence to convergent sequence?

(5) How is the strength of the A_t comparing to the identity matrix?

Notations and Background Materials

w= {the set of all complex sequences}

c= {the set of all convergent complex sequences}

$c(A)$ ={y: Ay $\in$ c}

l = {y: $\sum_{k=0}^{\infty} |y_k| < \infty$}

$l(A)$ ={y: Ay$\in l$ }

Regular Matrix

A matrix is regular if $lim_{n\to\infty}Z_n$= a$\Rightarrow$ $lim_{n\to\infty}(AX)_n$=a. That is a sequence Z is convergent to A$\Rightarrow$ the A-transform of Z also converses to a.

The Sliverman-Toeplitz Rule

We state the following famous Sliverman-Toeplitz Rule as Proposition I without proof and apply it.

Proposition I: A matrix A $=(a_{n,k})$ is regular if and only if

(i) $\lim_{n\to\infty} a_{n,k} = 0$ for each k= 0,1,…,

(ii) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} = 1,$ and

(iii) $\sup_n \{\sum_{k=0}^{\infty} |a_{n;k}|\} \le M < \infty$ for some $M > 0.$

The Main Results

Theorem 1: The Abel Matrix A_t is a regular matrix for all t.

Proof: We use proposition 1 to prove the theorem. Note that

(1) $\lim_{n\to\infty} a_{n,k} = \lim_{n\to\infty} (1-t_n)^k t_n = 0$

(2) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = \lim_{n\to\infty} \sum_{k=0}^{\infty} t_n{}^k (1-t_n) = \lim_{n\leftarrow\infty} (1-t_n) \sum_{k=0}^{\infty} t_n{}^k = \frac{1-t_n}{1-t_n} = 1$ and

(3) $Sup_n \sum_{k=0}^{\infty} a_{n,k} = 1$

Hence by Proposition I, the Abel Matrix A_t is a regular matrix.

Remark 1: The A_t matrix maps a bounded sequence into a convergent sequence as shown by the following example. This shows that the A_t matrix is stronger than the identity matrix or c(A) is larger than c.

Example 1: Consider the bounded sequence given by $x_k = (-1)^k$

Then $(A_t x)_n = (1-t_n)\sum_{k=0}^{\infty}(t_n)^k(-1)^k$

$$= (1-t_n)\sum_{k=0}^{\infty}(-t^n)^k$$

$$= (1-t_n)\frac{1}{1+t_n}$$

$(A_t x)_n = \dfrac{1-t_n}{1+t_n} \Rightarrow$ $\lim_{n\to\infty}(A_t x)_n = 0$; hence $A_t x \in c$

Remark 2: Thee A_t matrix maps also <u>a divergent sequence x</u> into a convergent sequence as shown by the following example.

Example 2: Consider the unbounded sequence given by $x_k = (-1)^k(k+1)$. Note that

$$(A_t x)_n = \sum_{k=0}^{\infty}(1-t_n)t_n^k(-1)^k(k+1)$$

$$= (1-t_n)\sum_{k=0}^{\infty}t_n^k(-1)^k(k+1)$$

$$= (1-t_n)\sum_{k=0}^{\infty}(-t_n)^k(k+1)$$

$$= \frac{1-t_n}{(1+t_n)^2}$$

Now , $\lim_{n\to\infty}(A_t x)_n = \lim_{n\to\infty}\dfrac{1-t_n}{(1+t_n)^2} = 0$

Hence $A_t x \in c.$

Definition: A matrix A is an x-y matrix if the image Au of u under the transformation A is in Y wherever u is in x.

Knopp-Lorentz

The Matrix A is an $\ell - \ell$ matrix if and only if there exists a number $M > 0$ such that for every k,

$$\sum_{n=0}^{\infty} |a_{nk}| \leq M.$$

Theorem 2: A_t is $\ell - \ell \Leftrightarrow (1-t) \in \ell$

Lemma 1:

$$A_t \ \ell - \ell \text{ matrix} \Rightarrow (1-t) \in \ell.$$

Proof: We use the Knopp-Lorentz Rule.

$$A_t \text{ is } \ell - \ell \Rightarrow \leq \sum_{n=0}^{\infty} |a_{nk}| \leq M \quad \text{for each k}$$

$$\Rightarrow \sum_{n=0}^{\infty} \left|(1-t_n)t_n^k\right| \leq M$$

$$\Rightarrow \sum_{n=0}^{\infty} |(1-t_n)| \leq M \quad \text{(for k=0)}$$

$$\Rightarrow (1-t) \in \ell$$

Lemma 2:

$$1-t \in \ell \Rightarrow A_t \text{ is an } \ell - \ell \text{ matrix}$$

Proof: We use the Knopp-Lorentz Rule

$$\sum_{n=0}^{\infty} |a_{nk}| = \sum_{n=0}^{\infty} \left|(1-t_n)t_n^k\right|$$

$$\leq \sum_{n=0}^{\infty} (1-t_n) \leq M \quad \text{for some M>0 as } (1-t) \in \ell.$$

Now Theorem 2 follows by Lemmas 1&2.

Corollary 1. If A_t is an l-l matrix and $0 < t_n < w_n < 1$, then A_w is also an l-l matrix.

Proof: $0<t_n<w_n<1 \Rightarrow (1-w_n)<(1-t_n)$ and hence the corollary follows by Theorem 1.

Corollary 2. A_t is an l-l matrix $\Leftrightarrow$ arcsint $\in l$

Proof: The corollary follows by Theorem 1 using the basic inequality

$x < \arcsin x < \frac{x}{\sqrt{1-x^2}}$ for 0<x<1.

Remark 3. An l-l A_t matrix maps a bounded sequence into l as shown by the following example. This shows that the A_t matrix is stronger than the identity matrix in the *l-l* setting or l(A) is larger than *l*.

Example 3.

Assume A_t matrix is an *l-l* and consider the bounded sequence given by $x_k = (-1)^k$. We want to show that $A_t x \in l.$

Then $$(A_t x)_n = (1-t_n)\sum_{k=0}^{\infty}(t_n)^k(-1)^k$$

$$= (1-t_n)\sum_{k=0}^{\infty}(-t^n)^k$$

$$= (1-t_n)\frac{1}{1+t_n}$$

$$\leq (1-t_n)$$

Now A_t matrix is l-*l* $\Rightarrow$ *(1-t)*$\in$ l , by Theorem 2, and hence $A_t x \in l.$

Remark 4: An l-l A_t matrix maps unbounded sequence into l as shown by the following example.

Example 4: Assume A_t matrix is *l-l* and consider the unbounded sequence given by

$x_k = (-1)^k (k+1)$. Note that

$$(A_t x)_n = \sum_{k=0}^{\infty}(1-t_n)t_n^k(-1)^k(k+1)$$

$$= (1-t_n)\sum_{k=0}^{\infty}t_n^k(-1)^k(k+1)$$

$$= (1-t_n)\sum_{k=0}^{\infty}(-t_n)^k(k+1)$$

$$= \frac{1-t_n}{(1+t_n)^2}$$

$$\leq (1-t_n)$$

Now A_t matrix is l-*l* $\Rightarrow$ *(1-t)*$\in$l , by Theorem 2, and hence $A_t x \in l.$

Remark 5: Every sequence x for which $|x_k|^{\frac{1}{k}} \leq 1$ belongs to l(A_t) provided A_t is an l-l matrix.

Example5. Let $x_n = (-3)^{n.}$ Then x is not in l(A). Note that $|x_k|^{\frac{1}{k}} = 3 > 1$

References

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- **Mulatu Lemma**, *Logarithmic and Abel-Type Transformations into Gw* Southeast Asian Bulletin of Mathematics;2010, Vol. 34 Issue 2, p299