The Sturm Liouville Problems with a random variable in Boundary Conditions

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Abstract

We discussed the Sturm-Liouville problems with random variable ξn involving in bound-ary conditions which represent a support coefficient for elastic rope. We have aconclusion that if the random variable in the boundary condition have convergence property, then the eigenvalues will also have a similar convergence property. We also give an asymptotic formula to approximate the large eigenvalues. This formula give an asymptotic relationship between eigenvalues and the support coefficient ξn when the eigenvalues are very large.

1 Introduction

1.1 Background of Sturm Liouville problem with boundary conditions

Sturm-Liouville problem plays an important role in math and physics theory. In past decades, many mathematicians discussed the relationship between the boundary conditions and the eigen-values [1, 2, 4]. In 1996, Kong proved that with separated boundary conditions, the eigenvalue functions of Sturm-Liouville problem would not only be continuous, but also be differentiable [2, 3]. In 2005, Zettle summarized different boundary conditions of Sturm-Liouville problems [4].

In this paper, we are discussing about the Sturm-Liouville problem with separated boundary conditions and study the relationship between the eigenvalue and a stochastic variable in boundary conditions.

1.2 Formulation of Sturm Liouville with stochastic boundary conditions

Consider differential equation

$$\varphi''(x) + \lambda\varphi(x) = 0 \quad (1)$$

with boundary conditions

$$\varphi(0) = 0 \qquad (2)$$

$$\varphi'(L) + \xi(w)\varphi(L) = 0 \qquad (3)$$

Here $x \in [0, L]$. This boundary condition has physics background: A elastic rope is fixed at one endpoint $x = 0$. Whether or not it is also fixed at the other endpoint, that depends on the support coefficient ξ. ($\xi \geq 0$ is a stochastic variable.) When $\xi > 0$, the rope is fixed at $x = L$, When $\xi = 0$, the rope is not fixed at $x = L$.
Assume equation 1 satisfy one of the following boundary conditions:

$$\varphi'(L) + \xi_1(w)\varphi(L) = 0$$

$$\varphi'(L) + \xi_2(w)\varphi(L) = 0 \quad (4)$$

......

$$\varphi'(L) + \xi_n(w)\varphi(L) = 0$$

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Here $\xi_1(w) \geq 0$, $\xi_2(w) \geq 0, \ldots$ $\xi_n(w) \geq 0\ldots$ are stochastic variables for $\forall n \in N$ the equation 1 together with any of these boundary values conditions will generate a Sturm- Liouville problem. Each $\xi_n(w)$ has infinitely many eigenvalues $\lambda_{ni}(w)$, $i = 1, 2, 3, \ldots \infty$.

If $\xi_1(w), \xi_2(w)\ldots\xi_n(w)$ convergent to $\xi(w)$ almost everywhere, in other words, the probability that $\lim_{n\to\infty} \xi_n(w) = \xi(w)$ is equal to 1. Written as

$$P\{w \in \omega : \lim_{n\to\infty} \xi_n(w) = \xi(w)\} = 1 \quad (5)$$

then

$$P\{w \in \omega : \lim_{n\to\infty} \lambda_{ni}(w) = \lambda_i(w)\} = 1 \quad (6)$$

In other words, each eigenvalues will have a similar convergence property.

2 Convergence Result of Eigenvalues

For $\forall n \in N$, any $\xi_n(w) \geq 0$, it is easy to see that for this kind of boundary value problem, only when eigenvalues $\lambda_n > 0$, non-trivial solutions exist. In equation 1, a general solution is given as

$$\varphi_n(x) = A_n \cos(x\sqrt{\lambda_n}) + B_n \sin(x\sqrt{\lambda_n}) \quad (7)$$

Because the 1st boundary condition $\varphi_n(0) = 0$, $\Rightarrow A_n = 0$. Because of the 2nd boundary condition,

$\varphi'_n(L) + \xi_n \varphi_n(L) = 0$

So

$\sqrt{\lambda_n} \cos(L\sqrt{\lambda_n}) + \xi_n \sin(L\sqrt{\lambda_n}) = 0$

If We get

$\tan(L\sqrt{\lambda_n}) = -\sqrt{\lambda_n}/\xi_n$

then

$\cot(L\sqrt{\lambda_n}) = -\xi_n/\sqrt{\lambda_n}$ (8)

If We let $x_n = L\sqrt{\lambda_n}$, then

$\cot(x_n) = -L\,\xi_n/x_n$ (9)

The Figure 1 gives us an illustration of intersection of functions y=cot(x) and y= $-L\,\xi_n/x_n$.

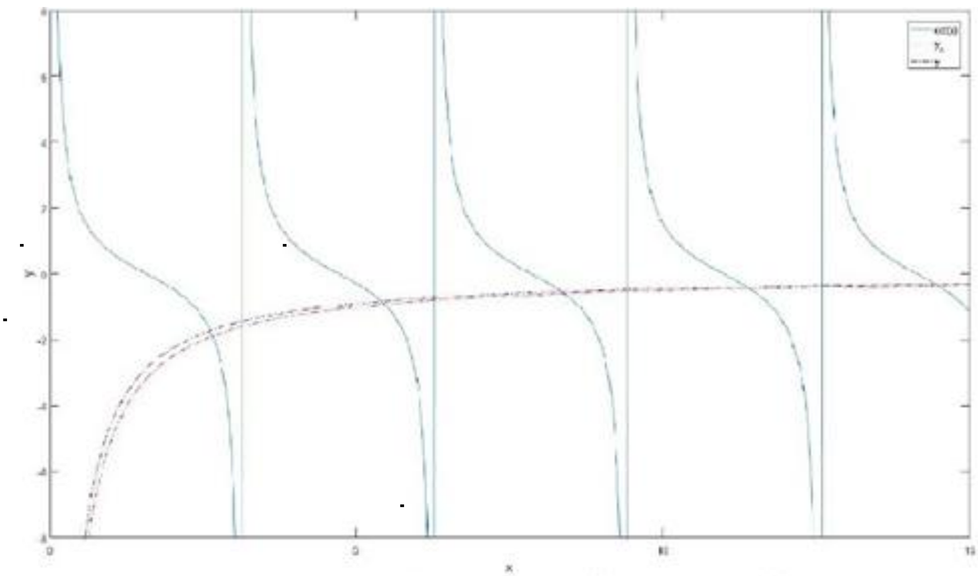

Figure 1: intersection of graph of cot(x) with $y_n = -\frac{L\xi_n}{x}$ and $y = -\frac{L\xi}{x}$: $\lim_{n\to\infty} \xi_n = \xi$
The i-th intersection point of $y_n = -\frac{L\xi_n}{x}$ with cot(x) is $(\bar{x}_i, \bar{y}_i)$, corresponding i-th eigenvalue is $\lambda_{ni} = (\frac{\bar{x}_i}{L})^2$. similarly, the i-th intersection point of $y = -\frac{L\xi}{x}$ with cot(x) is (x_i, y_i), corresponding i-th eigenvalue is $\lambda_i = (\frac{x_i}{L})^2$. Because of the continuity of $y = -\frac{L\xi}{x}$, when $\lim_{n\to\infty} \xi_n = \xi$, $\lim_{n\to\infty} \bar{x}_i = x_i$. So $\lim_{n\to\infty} \lambda_{ni} = \lambda_i$.

In summary, λ_{ni}, $i \in N$ are eigenvalues corresponding to boundary condition

$\varphi(0) = 0$ (10)
$\varphi'(L) + \xi_n(w)\varphi(L) = 0$ (11)

λ_i, $i \in N$ are eigenvalues corresponding to boundary condition

$\varphi(0) = 0$ (12)
$\varphi'(L) + \xi(w)\varphi(L) = 0$ (13)

If

$$P\{w \in \omega : \lim_{n\to\infty} \xi_n(w) = \xi(w)\} = 1$$

then

$$P\{w \in \omega : \lim_{n\to\infty} \lambda_{ni}(w) = \lambda_i(w)\} = 1$$

3 Asymptotic Analysis of large eigenvalues

By observation, we are able to make a more detailed asymptotic analysis of eigenvalues λ_{nk} when $k \to \infty$.

$\pi/2 < L\sqrt{\lambda_{n1}} < \pi$, $3\pi/2 < L\sqrt{\lambda_{n2}} < 2\pi$ Here $L\sqrt{\lambda_{nk}} - (k - 1/2)\pi \to 0$ as $k \to \infty$.

When k is large, we assume that

$$L\sqrt{\lambda_{nk}} = (k - 1/2)\pi + e(k) \quad (14)$$

$e(k) > 0$ is an error term that $\lim_{k\to\infty} e(k) = 0$.

Apply Taylor expansion of cot(x) around $x = (k - 1/2)\pi$ we get

$\cot((k - 1/2)\pi + e(k)) = -e(k) + O(e(k)^3)$ (15)

then as $e(k) \to 0$, substituting (14), (15) into (9):

$$- e(k)+O(e(k)^3)= - L\,\xi n/((k-1/2)\,\pi+ e(k)) \quad (16)$$

From (16) we derived that the error

$$- e(k)= - L\,\xi n/k\,\pi + O(1/k^2) \text{ as } k \to \infty. \quad (17)$$

So we have derived the following asymptotic formula

$$L\sqrt{\lambda_{nk}} = (k-1/2)\,\pi+ L\,\xi n/k\,\pi + O(1/k^2) \text{ as } k \to \infty. \quad (18)$$

References

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